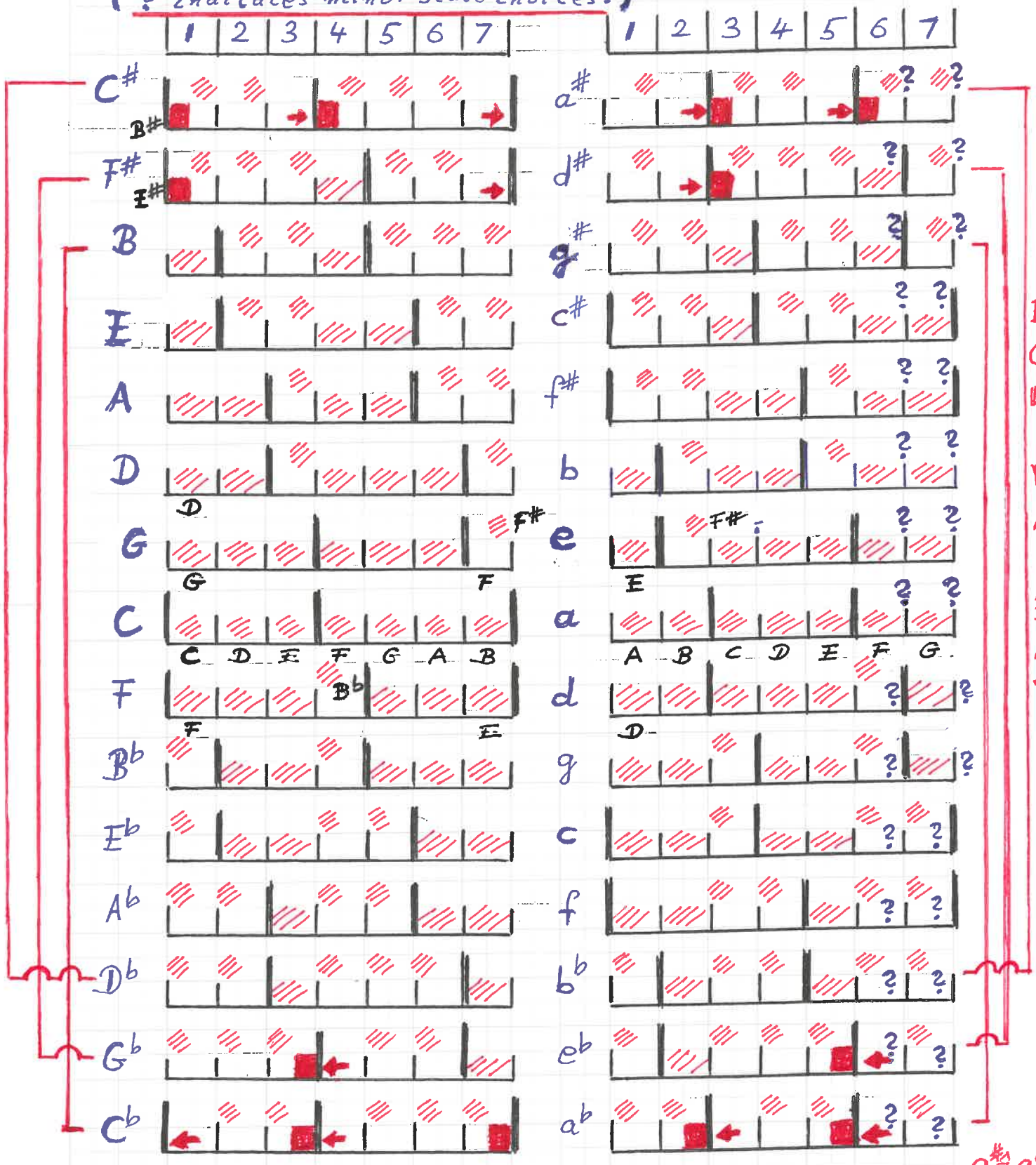
Key/Fingering Charts to Memorize

9/28/2024

The illustrations show the key patterns of all scales.
('?' indicates minor scale choices.)



$B \triangleq C^{\flat}$
$F^{\#} \triangleq G^{\flat}$
$C^{\#} \triangleq D^{\flat}$

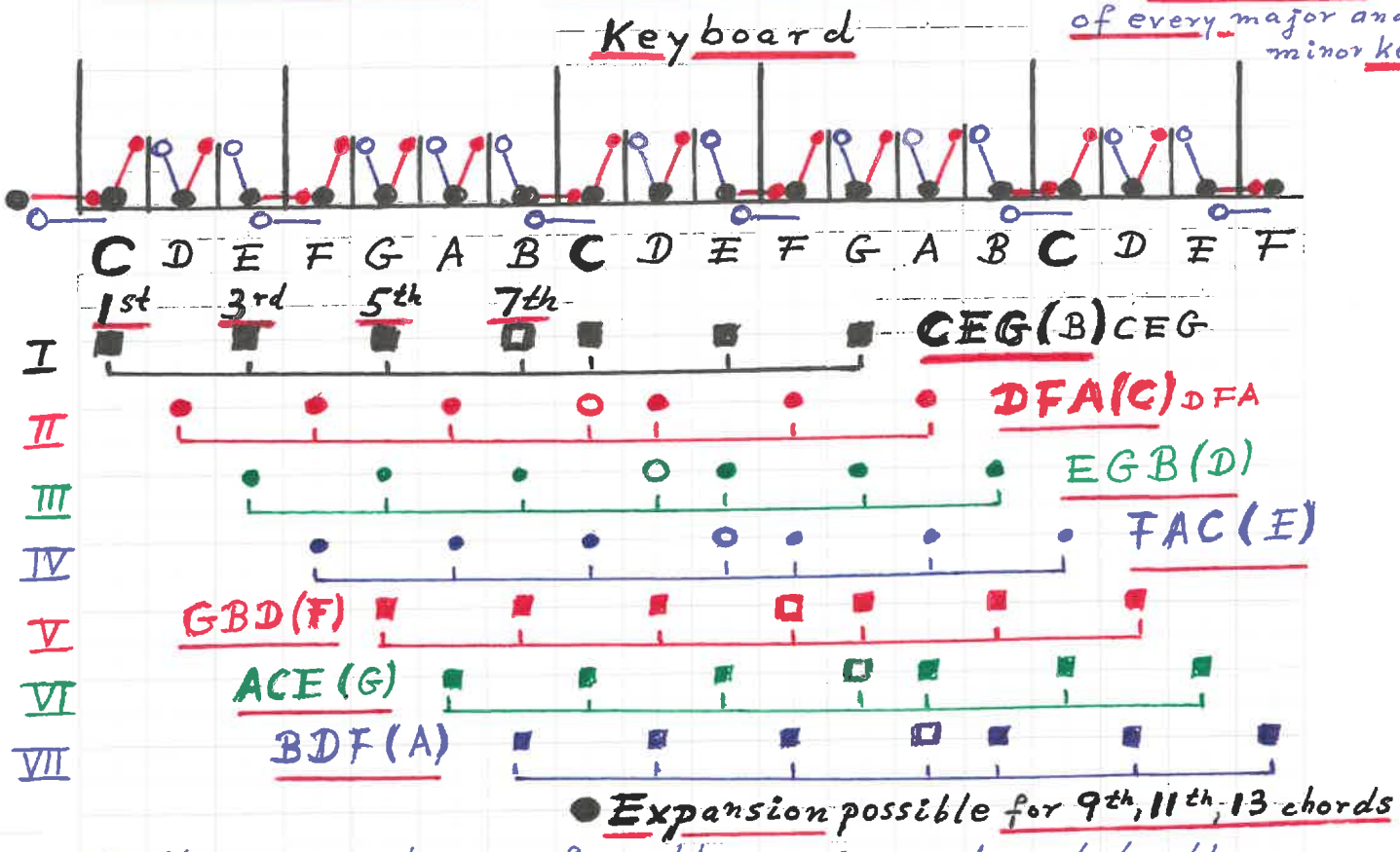
$g^{\#} \triangleq a^{\flat}$
$d^{\#} \triangleq e^{\flat}$
$a^{\#} \triangleq b^{\flat}$

→ One can think of all patterns as patterns of the white keys for C, D, E, F, G, A, B — where BLACK KEYS ARE VIEWED AS EXTENSIONS OF WHITE KEYS.

• Scales, Triads, 7th Chords, Inversions, Key-inherent Chords, ...

9/29/2024

• One must know the '#'s and 'b's of every major and minor key.



• Expansion possible for 9th, 11th, 13 chords

→ Memorization of patterns is reduced to the seven notes C, D, E, F, G, A, B (white keys only).

→ Each of the letters C, D, ..., B simultaneously represents its "three harmonic definitions, i.e., $C \hat{=} \{C^b, C, C^\#\}, \ldots, B \hat{=} \{B^b, B, B^\#\}$.

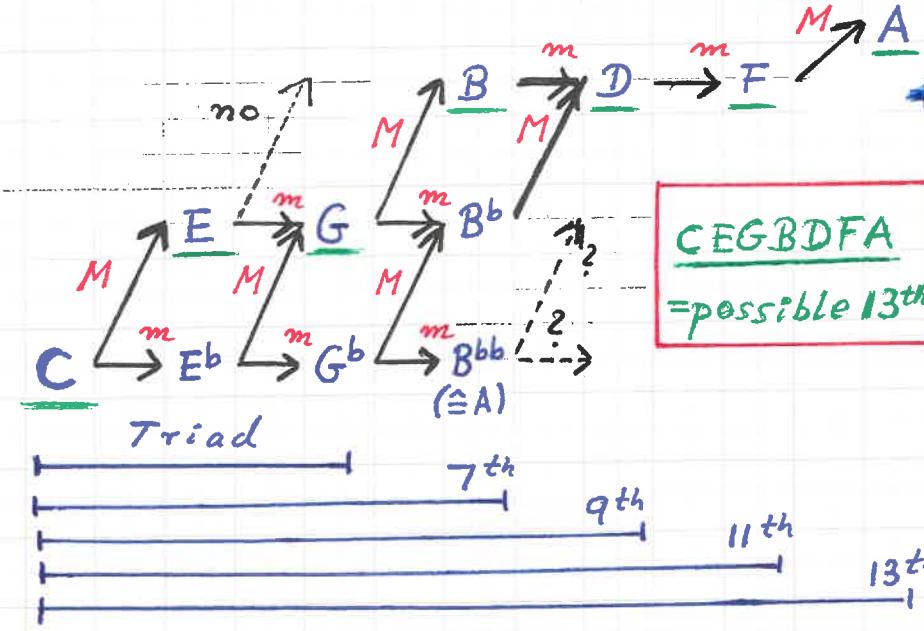
→ Considering necessary modifications of C, D, E, F, G, A, B via '#' or 'b' required by a specific key, are indicated in the keyboard illustration by / and — for '#' and ⚲ and o— for 'b' symbols, implying black keys.

→ The seven key-inherent triads and 7th chords (I – VII) are shown under the keyboard. For each of the shown triads and 7th chords, the 1st, 3rd, 5th and 7th notes are indicated — repeating the 1st, 3rd and 5th notes one octave higher for inversions: An example is CEGB $\xrightarrow{1}$ EGBC $\xrightarrow{2}$ GBCE $\xrightarrow{3}$ BCEG.

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Representation of 7th, 9th, 11th, 13th Chords 9/30/2024

➡ One can view these complex "jazz chords" as "stacks of 3rds." One stacks major and minor 3rds. In the following, a major 3rd chord is called M3rd or M3 or simply M; a minor 3rd chord is called m3rd or m3 or simply m. One can use a graph representation to capture all possible note stacks.



CEGBDFA
=possible 13th

➡ Using C as root note, for example, one can use the directed graph shown here to construct all possible chords obtained by stacking M3rds and m3rds. The figure indicates that some possible choices are (usually) not considered ("no") and that it is not immediately clear whether certain theoretical possibilities are "harmonically meaningful" ("?").

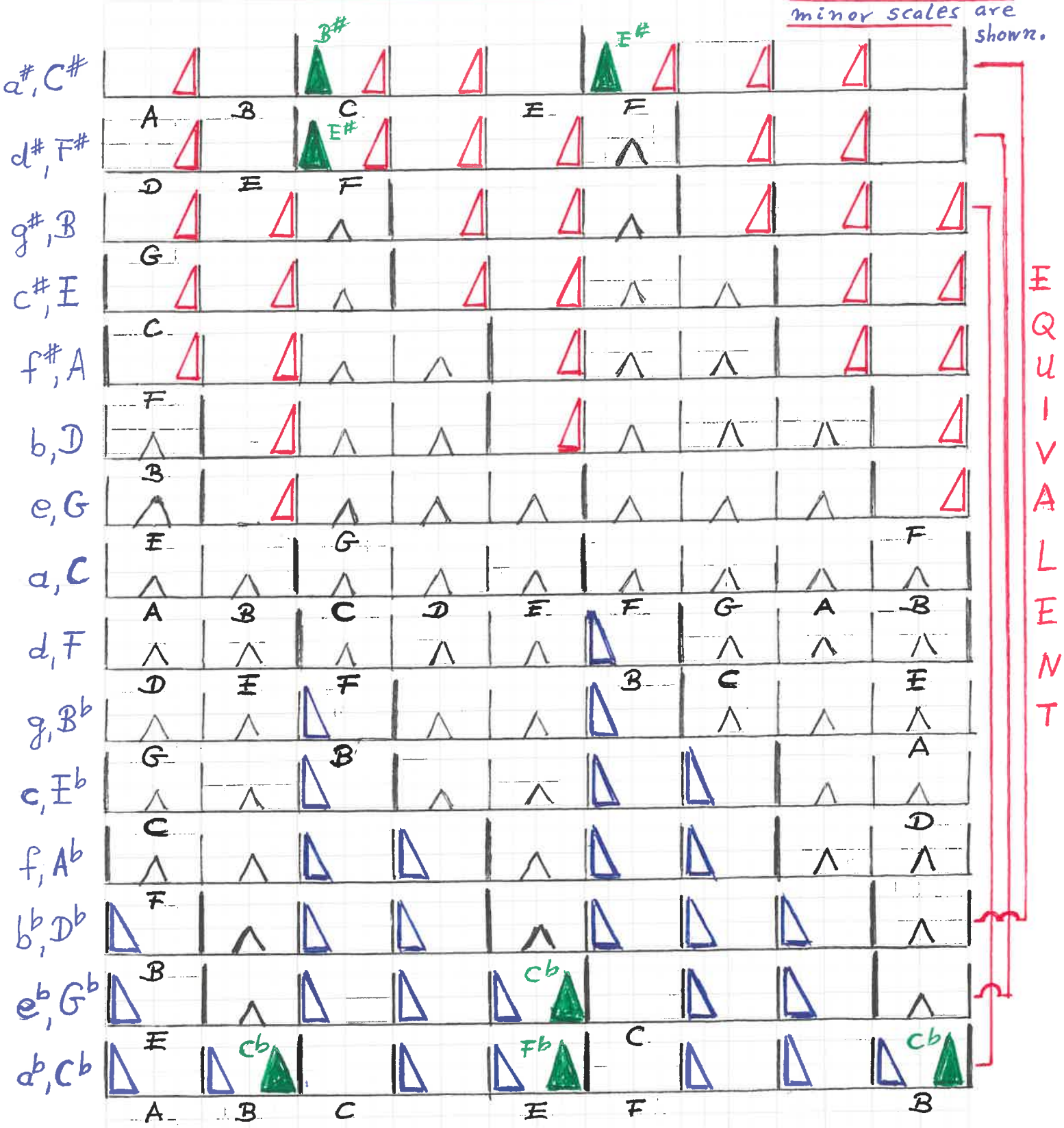
➡ The possible paths in the directed graph represent possible chords; one often does not (have to) play all notes of very complex chords. The underlined notes in the graph highlight the fact that all notes of a key are a 13th.

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• Major and Minor Scales — Represent with White Keys only!

9/30/2024

• Major and relative natural minor scales are shown.



• Principle: All major and minor scales are "considering" only the notes C, D, E, F, G, A, B. Key-defined '#' and 'b' are "thought of" as "white-key-associated parts." The figure uses four colored symbols to indicate keys to be played.

Scales and Chords – Various Thoughts

10/1/2024

Piano Keyboard – Numbered Keys

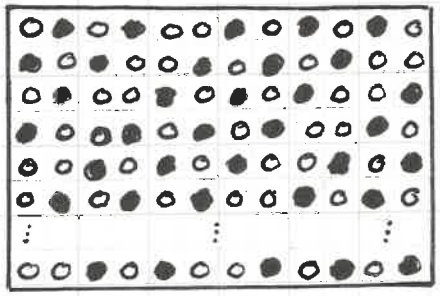
No. of keys: $3+7\cdot 12+1 = 88$



Chromatic Scales and Use of White / Black Keys:

1	2	3	4	5	6	7	8	9	10	11	12
C	C#	D	D#	E	F	F#	G	G#	A	A#	B
C#	D	D#	E	F	F#	G	G#	A	A#	B	C
D	D#	E	F	F#	G	G#	A	A#	B	C	C#
D#	E	F	F#	G	G#	A	A#	B	C	C#	D
E	F	F#	G	G#	A	A#	B	C	C#	D	D#
F	F#	G	G#	A	A#	B	C	C#	D	D#	E
F#	G	G#	A	A#	B	C	...			...	F
⋮											⋮
B	C	C#	D	D#	E	F	F#	G	G#	A	A#

The left table lists the "12 chromatic scales" – using as 1st note C, C#, D, ..., A, A#, B. The resulting 12 scales lead to a symmetric 12×12 "binary matrix" representing the corresponding use of white (○) and black (●) keys on the keyboard. The left figure illustrates this 12×12 "matrix of patterns."



Chord Notes Viewed as "Sets of Notes":

Often / usually, the lowest (highest) note played serves as bass (melody) note with its special function. Since the "notes in a set" do not reflect specific keys to be played, one can "pick" notes from the set to define a bass note or define inversions – "spreading out notes into both hands."

C used as bass note (left); "4 inversions" from set {C, E, G, B♭}

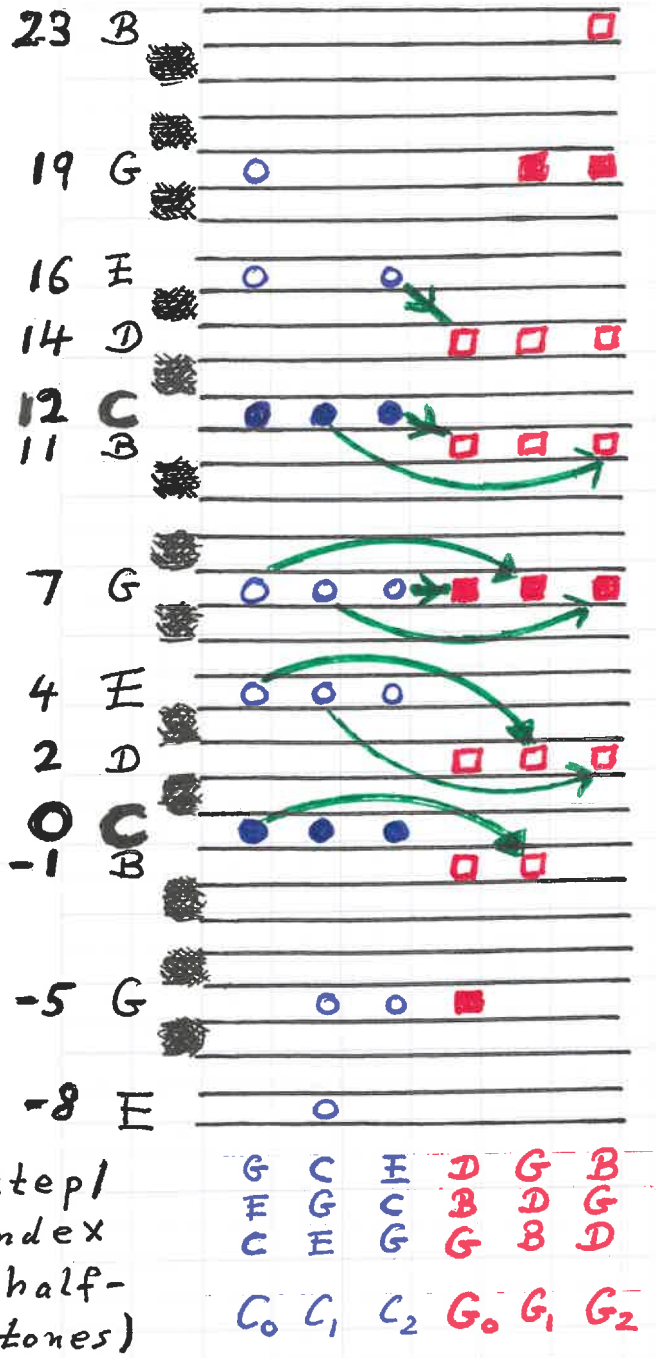
C	E	G	B♭
C	E	B♭	G
C	G	E	B♭
C	G	B♭	E
C	B♭	E	G
C	B♭	G	E

C	E	G	B
E	G	B	C
G	B	C	E
B	C	E	G

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Voice Leading & Optimization

10/11/2024



It is possible to use a simple mathematical approach to determine a "good voicing" when establishing a chord progression that moves from chord to chord (including chord inversions). The left figure shows C and G major chords in root and 1st and 2nd inversion positions, called C_0, C_1, C_2 and G_0, G_1, G_2, respectively. All chords are shown twice, shown "in two different octaves." The goal is to determine the pair(s) of a C and G major chord(s) defining the "smoothest" voicing(s) — indicated by the green arrows in the figure.

The figure shows the "keyboard vertically-1" uses one line per note in the 12-tone chromatic system, and uses integers as "indices" for the lines (with 0 being associated with [one of] the C notes):

$$C_0=\binom{7}{4}_0+i\binom{12}{12}_{12},\ C_1=\binom{12}{7}_4+i\binom{12}{12}_{12},\ C_2=\binom{16}{12}_7+i\binom{12}{12}_{12},\ i=\ldots,-1,0,1,\ldots,$$

and

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$G_0 = \begin{pmatrix}14\\11\\7\end{pmatrix} + j\begin{pmatrix}12\\12\\12\end{pmatrix}$, $G_1 = \begin{pmatrix}19\\14\\11\end{pmatrix} + j\begin{pmatrix}12\\12\\12\end{pmatrix}$, $G_2 = \begin{pmatrix}23\\19\\14\end{pmatrix} + j\begin{pmatrix}12\\12\\12\end{pmatrix}$, $j = \ldots -1, 0, 1, \ldots$

represent "all possible" C_0, C_1, C_2 and G_0, G_1, G_2 chords in all octaves available on the keyboard (and "beyond"). For example, the goal could be the "smoothest" moves/progressions (I-V) from $C_0 = \begin{pmatrix}7\\4\\0\end{pmatrix}$ to one of the G chords; from $C_1 = \begin{pmatrix}12\\7\\4\end{pmatrix}$ to one of the G chords; and from $C_2 = \begin{pmatrix}16\\12\\7\end{pmatrix}$ to one of the G chords. Therefore, one must define a simple "distance" for two chords — two triads in this scenario — to determine a pair of C and G major triads that are "closest" to each other, based on individual "note-to-note distances." For two triads $N_1 N_2 N_3$ and $n_1 n_2 n_3$ (with N_i and n_i being their notes) one can define a "distance" as the sum of the absolute values of corresponding note-pair distances, i.e., $\text{distance} = |N_1 - n_1| + |N_2 - n_2| + |N_3 - n_3|$, where the values of N_i and n_i are the note-associated integers shown in the figure on the previous page. For the three chords C_I, $I = 0, 1, 2$, and G_J, $J = 0, 1, 2$, one can thus compute $3 \times 3 = 9$ "distance" values $d_{I,J}$ — that define the minimal-distance chord G_J for a given chord C_I. Thus, we must calculate the values of $d_{0,J}$, $d_{1,J}$ and $d_{2,J}$ $(J = 0, 1, 2)$ for C_0, C_1, C_2 as defined above.

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• Note. For a given chord (C_0 or C_1 or C_2) one must first determine the closest chord (G_0, G_1, G_2) — from the set of all the considered G chords on the keyboard — to be able to determine the smallest values for $d_{0,0} = dist(C_0, G_0)$, $d_{0,1} = dist(C_0, G_1)$, ..., $d_{2,2} = dist(C_2, G_2)$. Assuming that these closest G chords are known on the keyboard for the specific chords C_0, C_1 and C_2, one can calculate the values of $d_{0,0}, \dots, d_{2,2}$ as described in the following.

$\Rightarrow$ Chord-to-chord distances $d_{I,J}$, $I,J = 0,1,2$:

$C_0 = \begin{pmatrix} 7 \\ 4 \\ 0 \end{pmatrix}$, $C_1 = \begin{pmatrix} 12 \\ 7 \\ 4 \end{pmatrix}$, $C_2 = \begin{pmatrix} 16 \\ 12 \\ 7 \end{pmatrix}$; $G_0 = \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix}$, $G_1 = \begin{pmatrix} 7 \\ 2 \\ -1 \end{pmatrix}$, $G_2 = \begin{pmatrix} 11 \\ 7 \\ 2 \end{pmatrix}$

$\Rightarrow d_{0,0} = |0+5| + |4+1| + |7-2| = 5+5+5 = 15$

$d_{0,1} = |0+1| + |4-2| + |7-7| = 1+2+0 = \mathbf{3}$ MIN

$d_{0,2} = |0-2| + |4-7| + |7-11| = 2+3+4 = 9$

$G_0 = \begin{pmatrix} 14 \\ 11 \\ 7 \end{pmatrix}$

$d_{1,0} = |4-7| + |7-11| + |12-14| = 3+4+2 = 9$

$d_{1,1} = |4+1| + |7-2| + |12-7| = 5+5+5 = 15$

$d_{1,2} = |4-2| + |7-7| + |12-11| = 2+0+1 = \mathbf{3}$ MIN

$d_{2,0} = |7-7| + |12-11| + |16-14| = 0+1+2 = \mathbf{3}$ MIN

$G_1 = \begin{pmatrix} 19 \\ 14 \\ 11 \end{pmatrix}$

$d_{2,1} = |7-11| + |12-14| + |16-19| = 4+2+3 = 9$

$d_{2,2} = |7-2| + |12-7| + |16-11| = 5+5+5 = 15$

$\Rightarrow \mathcal{D} = \begin{bmatrix} d_{0,0} & d_{0,1} & d_{0,2} \\ d_{1,0} & d_{1,1} & d_{1,2} \\ d_{2,0} & d_{2,1} & d_{2,2} \end{bmatrix} = \begin{bmatrix} 15 & 3 & 9 \\ 9 & 15 & 3 \\ 3 & 9 & 15 \end{bmatrix}$.

OPTIMAL:

C_0G_1: (0,4,7)(-1,2,7);

C_1G_2: (4,7,12)(2,7,11);

C_2G_0: (7,12,16)(7,11,14).

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One can summarize this result as follows:

1) Given the chord CEG "on steps 0, 4 and 7," the closest G triad is BDG "on steps -1, 2 and 7."

2) Given the chord EGC "on steps 4, 7 and 12," the closest G triad is DGB "on steps 2, 7 and 11."

3) Given the chord GCE "on steps 7, 12 and 16," the closest G chord is GBD "on steps 7, 11 and 14."

AND: The three calculated smallest-chord-distance pairs CEG-BDG, EGC-DGB and GCE-GBD all have the same minimal chord-to-chord (here, triad-to-triad) distance: **3**. In other words, these three pairs have a "summed up" note-to-note distance of only **3** halftones.

• As mentioned, for a given triad $N_1 N_2 N_3$ one must, in a "pre-step," determine the closest triad (on the keyboard) $n_1 n_2 n_3$ from the set of "octave-shifted versions" of $n_1 n_2 n_3$. One has the "fixed" triad $\begin{pmatrix} N_3 \\ N_2 \\ N_1 \end{pmatrix}$ and one must determine the/a "variable" triad $\begin{pmatrix} n_3 \\ n_2 \\ n_1 \end{pmatrix} + j \begin{pmatrix} 12 \\ 12 \\ 12 \end{pmatrix}$ that has minimal distance from the "fixed" triad. Again, N_i and n_i are integers (representing notes), and the value of j is the value to be optimally calculated. ONE CAN VIEW THE AVERAGE $(N_1+N_2+N_3)/3$ AS "THE MIDPOINT" OF THE FIXED TRIAD $\begin{pmatrix} N_3 \\ N_2 \\ N_1 \end{pmatrix}$. THE AVERAGE OF THE VARIABLE TRIAD $\begin{pmatrix} n_3+j\,12 \\ n_2+j\,12 \\ n_1+j\,12 \end{pmatrix}$ IS "THE MIDPOINT" $(n_1+n_2+n_3)/3 + 12j$. ONE MUST DETERMINE j.

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Thus, one determines the/an optimal value for j by minimizing the chord-to-chord — i.e., triad-to-triad — distance, given as $(N_1+N_2+N_3)/3 - \left((n_1+n_2+n_3)/3 + 12j\right) = (M-m) - 12j$. Here, $M=(N_1+N_2+N_3)/3$, $m=(n_1+n_2+n_3)/3$ and $j=\ldots,-1,0,1,\ldots$. Therefore, the goal is to "move" or "shift" the variable chord by an integer-multiple of 12 (12 halftones). MORE PRECISELY, ONE MUST DETERMINE THE INTEGER j FROM THE FINITE SET $\{\ldots,-2,-1,0,1,2,\ldots\}$ THAT MINIMIZES THE ABSOLUTE VALUE OF $(M-m)-12j$, i.e., THE VALUE OF $|(M-m)-12j| = |\Delta-12j|$, where $\Delta = M-m$ is the chords' midpoint distance.

Ex.: fixed chord $=\begin{pmatrix}7\\4\\0\end{pmatrix}$; variable chord $=\begin{pmatrix}14\\11\\7\end{pmatrix}+j\begin{pmatrix}12\\12\\12\end{pmatrix}$

$\Rightarrow (N_1,N_2,N_3)=(0,4,7)\,;\,(7+12j,\,11+12j,\,14+12j)$

$\Rightarrow M=(0+4+7)/3=11/3\,;\; m=32/3$

$\Rightarrow \Delta = 11/3 - 32/3 = -21/3 = -7$

"Shifting Go chord on keyboard" ⇒

j	-2	-1	0	1
$\lvert\Delta-12j\rvert$	$\lvert-7+24\rvert = 17$	$\lvert-7+12\rvert = 5$	$\lvert-7-0\rvert = 7$	$\lvert-7-12\rvert = 19$

OPTIMAL: $-j=-1$

⇒ Given the C_0 chord $\begin{pmatrix}7\\4\\0\end{pmatrix} \hat{=} \begin{pmatrix}G\\E\\C\end{pmatrix}$, the closest G_0 chord $\begin{pmatrix}14\\11\\7\end{pmatrix}+j\begin{pmatrix}12\\12\\12\end{pmatrix}$ is the chord $\begin{pmatrix}14\\11\\7\end{pmatrix}-1\begin{pmatrix}12\\12\\12\end{pmatrix}=\begin{pmatrix}2\\-1\\-5\end{pmatrix}\hat{=}\begin{pmatrix}D\\B\\G\end{pmatrix}$.

⇒ The resulting distance between C_0 and G_0 is therefore $d_{0,0}=|0+5|+|4+1|+|7-2|=15$.

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• **Note.** This "mathematically formalized" method for shifting a given "variable chord" by octaves on the keyboard is practically simple. For example, the 88-key keyboard can have the associated halftone numbering of keys as $A0 = -39, \ldots, C1 = -36, \ldots, C2 = -24, \ldots, C3 = -12, \ldots, C4 = 0, \ldots, C5 = 12, \ldots, C6 = 24, \ldots, C7 = 36, \ldots, C8 = 48$. Indeed, this numbering/indexing of all halftones refers to 88 notes: $48 - (-39) + 1 = 88$. Since the keyboard has only seven complete octaves, one can "easily see" the chord in root, 1st inversion or 2nd inversion position that is closest to a given chord — at least for simple chord pairs like triad pairs.

• **Note.** In order to calculate a "distance" for two chords ch1 and ch2 that do not consist of the same number of notes, one must generalize the concepts described above. For example, we use a 7th chord and a triad:

$$ch1 = (C_0^7) \begin{pmatrix} 10 \\ 7 \\ 4 \\ 0 \end{pmatrix} \hat{=} \begin{pmatrix} B^b \\ G \\ E \\ C \end{pmatrix}; \quad ch2 = (F_0) \begin{pmatrix} 12 \\ 9 \\ 5 \end{pmatrix} \hat{=} \begin{pmatrix} C \\ A \\ F \end{pmatrix}.$$

One can compute a distance d_{12} "from ch1 to ch2" and a distance d_{21} "from ch2 to ch1" by determining the closest note in ch2 (ch1) for a note in ch1 (ch2). Using the steps, one obtains $d_{12} = (5+1+2+1)/4 = 9/4$ and $d_{21} = (1+1+2)/3 = 4/3$ — normalizing by the nos. of notes.

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Thus, one could possibly use the (average) value $d = (d_{12} + d_{21})/2 = (9/4 + 4/3)/2 = 43/24$ as chord-to-chord distance, based on the individual chordnote-to-chordnote distances. Here, the value $43/24$ can be interpreted as an average halftone distance of less than 2.

• SMOOTHEST PROGRESSION OF THE EIGHT TRIADS DEFINING PACHELBEL'S CANON. This progression in the key of D major uses the progression I-V-vi-iii-IV-I-IV-V. The triads in root position are given as $DF^{\#}A$, $AC^{\#}E$, $BDF^{\#}$, $F^{\#}AC^{\#}$, GBD, $DF^{\#}A$, GBD and $AC^{\#}E$. One can now ask the following "combinatorial optimization question": Considering this chord progression, what is the optimal sequence of positions (root, 1st inversion or 2nd inversion positions) to be used when playing this progression to make it "sound optimally smooth in a harmonic sense"? Geometrically, one can understand the three positions of a triad as three points in 3D space (using as coordinates the halftone numbers/indices used above). Of all the combinatorically possible ways to establish triad positions (represented via points in 3D space) one must determine the "smoothest polyline in 3D space" connecting 8 "triad points." ⇒ OPTIMAL VOICING!

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One can view the problem of determining an optimal voicing of the chord/triad sequence I-V-vi-iii-IV-I-IV-V in the key of D major in Pachelbel's canon as a problem of determining the best positions (root, 1st inversion or 2nd inversion positions) for each of the 8 major and minor chords/triads when played in the defined order/sequence. When ignoring the specific octaves to be used for playing the triads in, the "position sets" for the triads involved are:

$$I=\left\{\begin{pmatrix}A\\F\#\\D\end{pmatrix},\begin{pmatrix}D\\A\\F\#\end{pmatrix},\begin{pmatrix}F\#\\D\\A\end{pmatrix}\right\};\quad V=\left\{\begin{pmatrix}E\\C\#\\A\end{pmatrix},\begin{pmatrix}A\\E\\C\#\end{pmatrix},\begin{pmatrix}C\#\\A\\E\end{pmatrix}\right\};$$

$$vi=\left\{\begin{pmatrix}F\#\\D\\B\end{pmatrix},\begin{pmatrix}B\\F\#\\D\end{pmatrix},\begin{pmatrix}D\\B\\F\#\end{pmatrix}\right\};\quad iii=\left\{\begin{pmatrix}C\#\\A\\F\#\end{pmatrix},\begin{pmatrix}F\#\\C\#\\A\end{pmatrix},\begin{pmatrix}A\\F\#\\C\#\end{pmatrix}\right\};$$

$$IV=\left\{\begin{pmatrix}D\\B\\G\end{pmatrix},\begin{pmatrix}G\\D\\B\end{pmatrix},\begin{pmatrix}B\\G\\D\end{pmatrix}\right\};\quad I=\ldots;\quad IV=\ldots;\quad V=\ldots\ .$$

$$\Rightarrow I=\{D_0,D_1,D_2\};\ V=\{A_0,A_1,A_2\};\ vi=\{b_0,b_1,b_2\};$$

$$iii=\{f\#_0,f\#_1,f\#_2\};\ IV=\{G_0,G_1,G_2\};\ I=\ldots;\ IV=\ldots;\ V=\ldots\ .$$

Further, one can set $D=D4\hat{=}2$ — and associate integers/indices with the notes of all triads (using, as done above, the definition $C4\hat{=}0$). Thus, all triads can be represented via integer vectors:

$$I=\left\{\begin{pmatrix}9\\6\\2\end{pmatrix},\begin{pmatrix}14\\9\\6\end{pmatrix},\begin{pmatrix}18\\14\\9\end{pmatrix}\right\};\quad V=\left\{\begin{pmatrix}16\\13\\9\end{pmatrix},\begin{pmatrix}21\\16\\13\end{pmatrix},\begin{pmatrix}25\\21\\16\end{pmatrix}\right\};$$

$$vi=\left\{\begin{pmatrix}18\\14\\11\end{pmatrix},\begin{pmatrix}23\\18\\14\end{pmatrix},\begin{pmatrix}26\\23\\18\end{pmatrix}\right\};\quad iii=\left\{\begin{pmatrix}13\\9\\6\end{pmatrix},\begin{pmatrix}18\\13\\9\end{pmatrix},\begin{pmatrix}21\\18\\13\end{pmatrix}\right\};$$

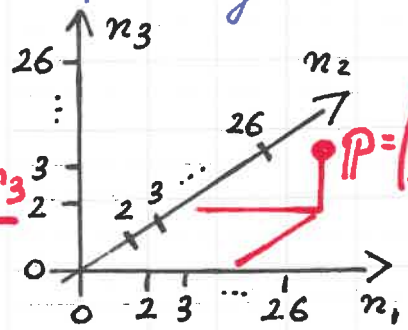
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$$\mathrm{IV} = \left\{ \begin{pmatrix} 14 \\ 11 \\ 7 \end{pmatrix}_1, \begin{pmatrix} 19 \\ 14 \\ 11 \end{pmatrix}_1, \begin{pmatrix} 23 \\ 19 \\ 14 \end{pmatrix} \right\}; \; \mathrm{I} = \ldots; \; \mathrm{IV} = \ldots; \; \mathrm{V} = \ldots \; .$$

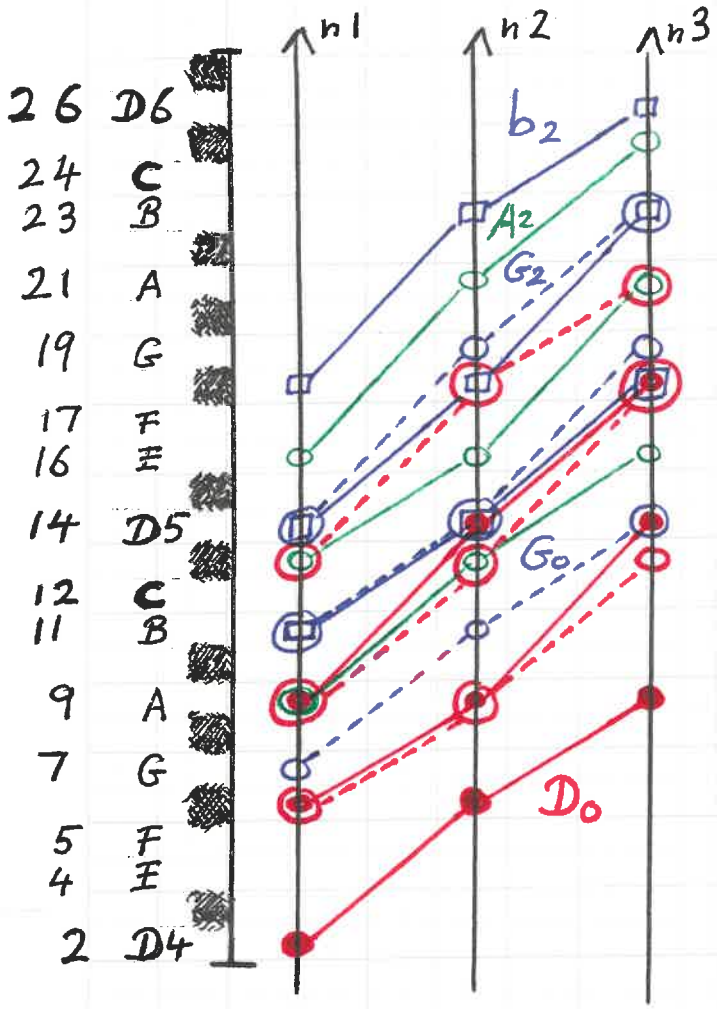
Based on this specific note-numbering, the lowest note (number) is D4 ($\hat{=}$ 2) and the highest note (number) is D6 ($\hat{=}$ 26) — i.e., these triads and their three positions are all viewed in the interval D4 – D6, defining 2 octaves.



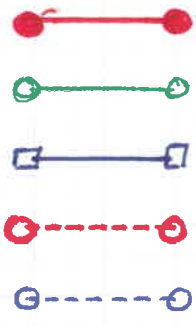
One can represent the three positions of the (sequence of) eight triads as "points" in a 3D space, using three integer coordinate axes n_1, n_2 and n_3 for the three stacked notes of a triad $(n_3, n_2, n_1)^T$, with n_1 being the lowest and n_3 being the highest note, see above figure. A point $\mathbb{P}$ is thus a "note-point" with integer coordinates n_1, n_2 and n_3: $\mathbb{P} = (n_1, n_2, n_3)^T$. Since the considered progression consists of eight triads, the progression can be thought of as a POLY-LINE CONNECTING EIGHT "NOTE-POINTS" (VERTICES) $\mathbb{P}_{\mathrm{I}}$, $\mathbb{P}_{\mathrm{V}}$, $\mathbb{P}_{\mathrm{vi}}$, $\mathbb{P}_{\mathrm{iii}}$, $\mathbb{P}_{\mathrm{IV}}$, $\mathbb{P}_{\mathrm{I}}$, $\mathbb{P}_{\mathrm{III}}$ and $\mathbb{P}_{\mathrm{V}}$. EACH OF THESE EIGHT POINTS IS A POINT THAT REPRESENTS ONE OF THREE TRIAD POSITIONS. Therefore, the number of possible 8-vertex polylines is $3^8 = 6561$. THE/AN OPTIMAL TRIAD PROGRESSION (POLY-LINE) MINIMIZES OVERALL NOTE-TO-NOTE MOVES.

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10/14/2024



Parallel coordinates visualization of all triads.



D_0, D_1, D_2 I
A_0, A_1, A_2 V
b_0, b_1, b_2 vi
$f^\#_0, f^\#_1, f^\#_2$ iii
G_0, G_1, G_2 IV

The left figure uses a parallel-coordinates visualization for the three notes of all triads, considering all three positions for each triad. Geometrically, one must determine a set of eight polylines, satisfying the Pachelbel progression, that minimizes the value of the sum of all "up" and "down" movements of notes. One can employ a computational approach and calculate this sum of note movements for the 6561 possibilities to determine the/an optimal triad progression.

A triad progression "in the middle" of D4-D6 is

F#	E	F#	F#	G	F#	G	A
D	C#	D	C#	D	D	D	E
A	A	B	A	B	A	B	C#

(A4 → A5)

$\hat{=} D_2\text{-}A_0\text{-}b_0\text{-}f^\#_1\text{-}G_1\text{-}D_2\text{-}G_1\text{-}A_1$.

2	2	0	1	1	1	2	
1	1	1	1	0	0	2	
0	2	2	2	2	2	2	

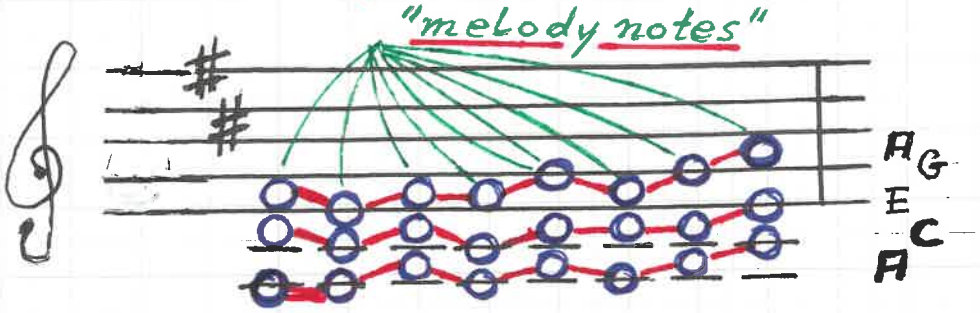
halftone ⇒ distances

Note-to-note halftone steps.

This progression only uses notes between A4 ($\hat{=}9$) and A5 ($\hat{=}21$). Note distances are ≤ 2.

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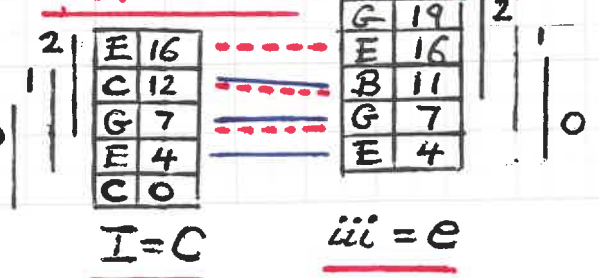
$I_2\ V_0\ vi_0\ iii_1\ IV_1\ I_2\ IV_1\ V_1$

$D_2\ A_0\ b_0\ f\#_1\ G_1\ D_2\ G_1\ A_1$

The left figure shows the chosen 8-triad progression that represents the Pachelbel progression. Again, this selection of 8 triads — with all triad notes in an octave-interval A-A — is just one of 6561 possibilities. It is not clear, i.e., it has not been computationally determined, whether this specific 8-triad progression in the key of D major is optimal in the sense of best-possible voice leading; nevertheless, one can "see" that all note-to-note movements when progressing from triad to triad are very small, i.e., less than 3 halftone steps (red line segments connecting notes).

• **Note.** The progression shown above involves a total of 27 halftone steps (using positive, absolute values of up- and down-movements only). Since this progression involves 21 (= 7×3) note-to-note movements, the average movement is $27/21 = 9/7 = 1\,2/7$ [steps].

Alternative for showing movement: 5-tuples



The left figure provides a visualization of C maj. and e min. triads, including all positions (0, 1, 2) and note numbers.

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{• Note. One can employ REAL-valued minimization methods to determine closest "position" of a triad (= the octave in which it is played) to a given "fixed" triad. This is a specific example:

$$C_0 = \begin{pmatrix} G \\ E \\ C \end{pmatrix} \hat{=} \begin{pmatrix} 7 \\ 4 \\ 0 \end{pmatrix} ; \; G_0(j) = \begin{pmatrix} 14 \\ 11 \\ 7 \end{pmatrix} + \begin{pmatrix} 12j \\ 12j \\ 12j \end{pmatrix} \hat{=} \begin{pmatrix} D \\ B \\ G \end{pmatrix}$$

⇒ One must determine the j-value that "moves" G_0 as close as possible to C_0.

⇒ Use the "sum of squared note-to-note differences" and minimize it:

$$d^2 = (14+12j-7)^2 + (11+12j-4)^2 + (7+12j-0)^2$$
$$= 3(7+12j)^2 = 147 + 504j + 432j^2$$

$$\Rightarrow (d^2)' = 504 + 864j \doteq 0 \Rightarrow j = -7/12 .$$

One determines the (signed) integer J that is closest* to $j = -7/12$: $J = -1$. *floor: $\lfloor -7/12 \rfloor = -1$

⇒ The G_0 chord/triad closest to C_0 as given is the triad $G_0^{opt} = G_0(-1) =$

$$= \begin{pmatrix} 14 \\ 11 \\ 7 \end{pmatrix} + \begin{pmatrix} -12 \\ -12 \\ -12 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix} \hat{=} \begin{pmatrix} D \\ B \\ G \end{pmatrix} = G_0 .$$

→ One can generalize this method for arbitrary triads:

$$ch1 = \begin{pmatrix} N_2 \\ N_1 \\ N_0 \end{pmatrix} ; \; ch2(j) = \begin{pmatrix} n_2+12j \\ n_1+12j \\ n_0+12j \end{pmatrix} \Rightarrow d^2 = \sum_{i=0}^{2} (n_i - N_i + 12j)^2 =$$

$$= \ldots = (\Delta_0^2 + \Delta_1^2 + \Delta_2^2) + 24(\Delta_0 + \Delta_1 + \Delta_2)j + 432j^2 , \; \Delta_i = n_i - N_i$$

$$\Rightarrow (d^2)' = 24(\Delta_0 + \Delta_1 + \Delta_2) + 864j \doteq 0 \Rightarrow j = -(\Delta_0 + \Delta_1 + \Delta_2)/36$$

By defining $\Delta = \Delta_0 + \Delta_1 + \Delta_2$, one obtains the optimal j-value as $j = -\Delta/36$. For the above example, this value is the value $-(7-0+11-4+14-7)/36 = -7/12$. }

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{• Note. Analogously, one can use REAL-value minimization to determine the closest "position" of three triads — the first triad being "fixed" and the second and third triads being "movable" by octaves. The three triads are:

$$ch1 = \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix};\quad ch2 = \begin{pmatrix} b_2 + 12j \\ b_1 + 12j \\ b_0 + 12j \end{pmatrix};\quad ch3 = \begin{pmatrix} c_2 + 12k \\ c_1 + 12k \\ c_0 + 12k \end{pmatrix}.$$

For example, one can add the "sums of squared note-to-note differences" for ch1 and ch2 and for ch2 and ch3 and minimize it:

$$d_{12}^2 = \sum_{i=0}^{3} (b_i - a_i + 12j)^2 = \sum_{i=0}^{3} (u_i + 12j)^2 =$$
$$= (u_0^2 + u_1^2 + u_2^2) + 24(u_0 + u_1 + u_2)j + 432j^2;$$
$$d_{23}^2 = \sum_{i=0}^{3} (c_i - b_i + 12(k-j))^2 = \sum_{i=0}^{3} (v_i + 12(k-j))^2 =$$
$$= (v_0^2 + v_1^2 + v_2^2) + 24(v_0 + v_1 + v_2)(k-j) + 432(k-j)^2$$
$$= (v_0^2 + v_1^2 + v_2^2) + 24(v_0 + v_1 + v_2)(k-j) + 432(k^2 - 2kj + j^2).$$

$$\Rightarrow d_{12}^2 = CON_{12} + 24uj + 432j^2;$$
$$d_{23}^2 = CON_{23} + 24v(k-j) + 432(k^2 - 2kj + j^2).$$

$$\Rightarrow D = d_{12}^2 + d_{23}^2 = CON + 24(u-v)j + 24vk + 864j^2 + 432k^2 - 864kj.$$

$$\Rightarrow \partial/\partial j\, D = 24(u-v) + 1728j - 864k \stackrel{!}{=} 0$$
$$\partial/\partial k\, D = 24v - 864j + 864k \stackrel{!}{=} 0$$

⇒ The following linear system must be solved:

$$72j - 36k = v - u \text{ and } 36j - 36k = v \Leftrightarrow$$
$$\Leftrightarrow 2j - k = (v-u)/36 \text{ and } j - k = v/36.$$
$$\Rightarrow j = -u/36;\quad k = -(u+v)/36.$$
$$\Rightarrow J = \text{ROUND}\lfloor j \rfloor;\quad K = \text{ROUND}\lfloor k \rfloor. \quad (\lfloor\,\rfloor = \text{floor})$$
$$\Rightarrow ch2_{opt} = \begin{pmatrix} b_2 + 12J \\ b_1 + 12J \\ b_0 + 12J \end{pmatrix};\quad ch3_{opt} = \begin{pmatrix} c_2 + K12 \\ c_1 + K12 \\ c_0 + K12 \end{pmatrix}.$$

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Example: We consider the 3-triad progression C_o (I), G_o (V) and F_o (IV) in the key of C:

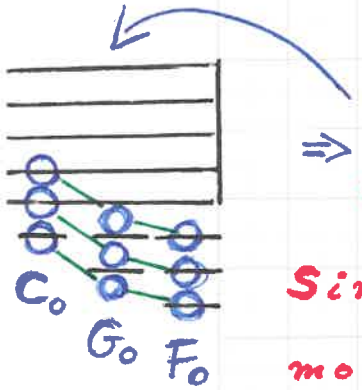
$$ch1=\begin{pmatrix}G\\E\\C\end{pmatrix}\hat{=}\begin{pmatrix}7\\4\\0\end{pmatrix};\quad ch2=\begin{pmatrix}D\\B\\G\end{pmatrix}\hat{=}\begin{pmatrix}14+12j\\11+12j\\7+12j\end{pmatrix};\quad ch3=\begin{pmatrix}C\\A\\F\end{pmatrix}\hat{=}\begin{pmatrix}0+12k\\-3+12k\\-7+12k\end{pmatrix}$$

$$\Rightarrow u=u_0+u_1+u_2=(7-0)+(11-4)+(14-7)=21$$

$$v=v_0+v_1+v_2=(-7-7)+(-3-11)+(0-14)=-42$$

$$\Rightarrow j=-u/36=-21/36=-7/12 \Rightarrow J=-1=\lfloor -7/12\rfloor$$

$$k=-(u+v)/36=21/36=7/12 \Rightarrow K=0=\lfloor 7/12\rfloor$$



$$\Rightarrow ch2_{opt}=\begin{pmatrix}D\\B\\G\end{pmatrix}\hat{=}\begin{pmatrix}14-12\\11-12\\7-12\end{pmatrix}=\begin{pmatrix}2\\-1\\-5\end{pmatrix};\quad ch3_{opt}=\begin{pmatrix}C\\A\\F\end{pmatrix}\hat{=}\begin{pmatrix}0+0\\-3+0\\-7+0\end{pmatrix}=\begin{pmatrix}0\\-3\\-7\end{pmatrix}$$

Since 4-chord / 4-triad progressions are the most commonly used progressions, we summarize the REAL-value minimization steps for 4 triads:

$$ch1=\begin{pmatrix}a_2\\a_1\\a_0\end{pmatrix};\quad ch2=\begin{pmatrix}b_2+12j\\b_1+12j\\b_0+12j\end{pmatrix};\quad ch3=\begin{pmatrix}c_2+12k\\c_1+12k\\c_0+12k\end{pmatrix};\quad ch4=\begin{pmatrix}d_2+12L\\d_1+12L\\d_0+12L\end{pmatrix}.$$

$$\Rightarrow d_{12}^2=(u_0^2+u_1^2+u_2^2)+24(u_0+u_1+u_2)j+432j^2.$$

$$d_{23}^2=(v_0^2+v_1^2+v_2^2)+24(v_0+v_1+v_2)(k-j)+432(k-j)^2.$$

$$d_{34}^2=(w_0^2+w_1^2+w_2^2)+24(w_0+w_1+w_2)(\ell-k)+432(\ell-k)^2.$$

$$\Rightarrow D=d_{12}^2+d_{23}^2+d_{34}^2=CON+24(uj+v(k-j)+w(\ell-k))$$
$$+432(j^2+(k-j)^2+(\ell-k)^2)$$
$$=CON+24((u-v)j+(v-w)k+w\ell)+432(2j^2+2k^2+\ell^2-2jk-2k\ell).$$

$$\Rightarrow \partial/\partial j\ D=24(u-v)+1728j-864k \doteq 0$$

$$\partial/\partial k\ D=24(v-w)-864j+1728k-864\ell \doteq 0$$

$$\partial/\partial \ell\ D=24w-864k+864\ell \doteq 0$$

$\Rightarrow$ The resulting linear system to be solved is:

$$2j-k=(v-u)/36 \text{ and } -j+2k-\ell=(w-v)/36 \text{ and } -k+j=-w/36$$

$$\Rightarrow j=-u/36;\ k=-(u+v)/36;\ \ell=-(u+v+w)/36$$

$$\Rightarrow J=\text{ROUND}[j];\ K=\text{ROUND}[k];\ L=\text{ROUND}[\ell].$$

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Example: We consider the following progression:

$$C_0 \hat{=} \begin{pmatrix} 7 \\ 4 \\ 0 \end{pmatrix};\ G_0 \hat{=} \begin{pmatrix} 14+12J \\ 11+12J \\ 7+12J \end{pmatrix};\ F_0 \hat{=} \begin{pmatrix} 0+12K \\ -3+12K \\ -7+12K \end{pmatrix};\ a_0 \hat{=} \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} + \begin{pmatrix} 12L \\ 12L \\ 12L \end{pmatrix} = \begin{pmatrix} d_2+12L \\ d_1+12L \\ d_0+12L \end{pmatrix}$$

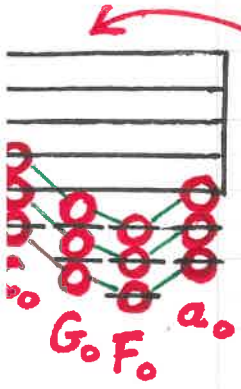
$\Rightarrow u = 21\ ;\ v = -42\ ;\ w = (-3+7)+(0+3)+(4-0) = 11$

$\Rightarrow j = -21/36 = -7/12\ ;\ k = -(21-42)/36 = 7/12\ ;$

$\ell = -(21-42+11)/36 = 5/18.$

$\Rightarrow J = -1\ ;\ K = 0\ ;\ L = 0\ \ (J=\lfloor -7/12 \rfloor;\ K=\lfloor 7/12 \rfloor;\ L=\lfloor 5/18 \rfloor)$

$\Rightarrow C_0 \hat{=} \begin{pmatrix} 7 \\ 4 \\ 0 \end{pmatrix};\ G^{\circ}_{opt} \hat{=} \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix};\ F^{\circ}_{opt} = \begin{pmatrix} 0 \\ -3 \\ -7 \end{pmatrix};\ a^{\circ}_{opt} \hat{=} \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}.$



This 4-triad progression of root-position triads C maj., G maj., F maj. and a min. is optimal: The total sum of up- and down-movements of corresponding notes of triad "neighbors" in the progression I-V-IV-vi is minimal (counting up- and down-movemnts of notes always as absolute – not signed – values). }

• **Note.** One can use this method for determining the "optimal" 8-triad progression, treated as an optimal voice leading problem, as follows: **1)** The first triad is given in a fixed location (in root position or in 1st or 2nd inversion). The other **7** triads are given in the order they should be played, considering all three possible positions for all **7** triads. Thus, the number of possible "triad picks" is $3^7 = 2187$. **2)** For each of these **2187** possibilities one determines the optimal "octave-placement." **3)** The best of these optimally placed triads wins.

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⊛ Issue: Minimization of sum of all possible triad-to-triad distances vs. minimization of sum of triad-to-next-triad distances!

• Note. ⊛ "CORRECTION." When defining the distance measure $\mathcal{D}$ (to be minimized) for a 3-triad (4-triad) triad progression as discussed, one "must" CONSIDER ALL 3 (6) POSSIBLE TRIAD PAIR DISTANCES, i.e., d_{12}^2, d_{13}^2 and d_{23}^2 (d_{12}^2, d_{13}^2, d_{14}^2, d_{23}^2, d_{24}^2 and d_{34}^2). Thus, for the 3-triad progression minimization, we also can/"must" use $d_{13}^2 = \sum_{i=0}^{3}((c_i - a_i)+12k)^2 = \sum_{i=0}^{3}(w_i+12k)^2$.

The properly expanded expression for $\mathcal{D}$ becomes:

$$\mathcal{D}^2 = d_{12}^2 + d_{13}^2 + d_{23}^2 = CON + 24(u-v)j + 24(v+w)k + 864j + 864k^2 - 864jk$$

(since $d_{13}^2 = (w_0^2+w_1^2+w_2^2) + 24(w_0+w_1+w_2)k + 432k^2$ $= CON + 24wk + 432k^2$).

$$\Rightarrow \frac{\partial}{\partial j}\mathcal{D} = 24(u-v) + 1728j - 864k \stackrel{!}{=} 0$$
$$\frac{\partial}{\partial k}\mathcal{D} = 24(v+w) - 864j + 1728k \stackrel{!}{=} 0 .$$

⇒ The resulting linear system to be solved is:

$$(u-v) + 72j - 36k = 0 \;\wedge\; (v+w) - 36j + 72k = 0$$
$$\Leftrightarrow\; 2j - k = (v-u)/36 \;\wedge\; -j + 2k = (v+w)/36 .$$

⇒ The (real-valued) solution to this system is:

$$j = (-2u+v-w)/108 \,,\quad k = (-u-v-2w)/108 .$$

Example. Given the three triads

$$C_0 \hat{=} \begin{pmatrix}7\\4\\0\end{pmatrix};\quad G_0 \hat{=} \begin{pmatrix}14+12J\\11+12J\\7+12J\end{pmatrix};\quad F_0 \hat{=} \begin{pmatrix}0+12K\\-3+12K\\-7+12K\end{pmatrix}$$ (see p. 14/15),

the additional value $w = (-7-0)+(-3-4)+(0-7) = -21$ leads to the new values for j and k:

$$j = (-42-42+21)/108 = -\frac{7}{12};\quad k = (-21+42+42)/108 = \frac{7}{12}$$

$$\Rightarrow J = \text{ROUND}(-7/12) = -1;\quad K = \text{ROUND}(7/12) = 1 \Rightarrow G_0 \hat{=} \begin{pmatrix}2\\-1\\-5\end{pmatrix};\quad F_0 \hat{=} \begin{pmatrix}12\\9\\5\end{pmatrix}.$$

or: $J = \lfloor -7/12 \rfloor = -1$; $K = \lfloor 7/12 \rfloor = 0 \Rightarrow G_0 = (-5,-1,2)^T$; $\tilde{F}_0 = (-7,-3,0)^T$.

