Harmonic Structures of Western Music: Thoughts on Representations Using Concepts from Algebra, Combinatorics, Geometry and Graph Theory

Note and Chord Graphs, the Tonnetz and Tree Structures, Tessellations, Dual Complexes, and Projections

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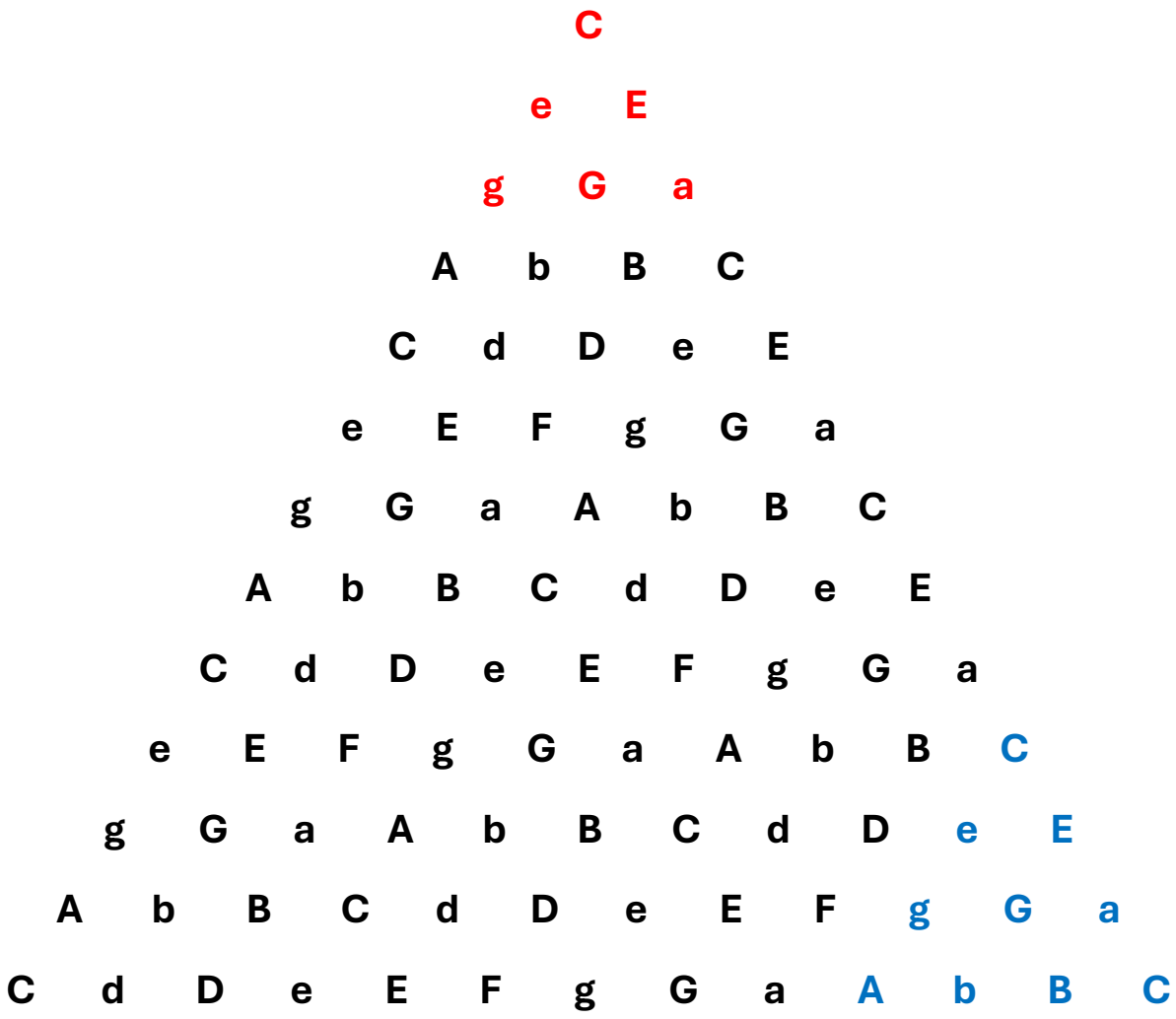
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Remark. *These notes were compiled by a person without formal training in music theory. The unifying goal of the notes is the objective of understanding structures that underly the complexity of harmony. The Notes do not use the proper and standard notations, and they lack the strict definition and consistent use of terms and mathematical rigor expected from a scientific manuscript. Exposing ideas in straightforward ways is the style used in these Notes. When technical terms are used, they are often explained by a simple sketch. Exact definitions and theorems from algebra, combinatorics, geometry and graph theory are not provided, as they are not needed for the purpose of this sketch, i.e., sharing simple and intuitive views on harmony. The notes should stimulate questions and thoughts.*

Introduction. These notes cover concepts for the characterization of harmonic relationships between keys/scales, chords and notes. A main goal is the use of structures and methods from geometry, graph theory and algebra to describe harmony and progressions. The first structure used to represent notes is the shown binary tree. Using the note C as tree root, one moves left to e=Eb via a minor 3rd (three semi-ones) and right to E via a major 3rd (four semi-tones).

Notation: d = Db = C#, e = Eb = D#, g = Gb = F# a = Ab = G#, b = Bb = A#.

C=0, d=1, D=2, e=3, E=4, F=5, g=6, G=7, a=8, A=9, b=10, B=11.



This tree representation built from the root note C is another way to construct the Tonnetz graph originally conceived by Leonhard Euler, see Euler [4]. Other studies closely related to the material discussed here are the works by Berg et al. [1], Cohn [2], Coxeter [3], Imai et al. [5], Rameau [6], Riemann [7], Rietsch [8], Tucker [9], Tymoczko [10] and Warren and Weimer [11]. Euler's Tonnetz and its generalizations provide a basis for representing and visualizing notes as well as major and minor triads of all keys/scales. Any note can serve as root of the binary tree construction discussed here.

One might ask "What is the purpose of a mathematical, geometrical characterization of notes, chords, keys, progressions etc.?" There are several ways to answer this question. On a high level, the two most important answers can be phrased as follows:

> **(1)** In the context of performing a harmonic analysis of a composition, it is helpful to have access to a general, fundamental scheme that characterizes all possible harmonic relationships between keys/scales, chords and notes. By relating the harmonic analysis of a piece of music to this fundamental harmony scheme, one can comprehend more fully the underlying compositional principles that a composer employed when writing the composition.
> **(2)** When one intends to improvise or compose oneself, and when one would like to utilize harmonic progressions that support a "smooth transition" when transitioning from one harmony to another or when involving modulations, knowing a general description of possibly ways to "move around" in a specific family of keys/scales or harmonies can provide the

needed manual for devising smooth movements between notes, chords and keys/scales.

Local neighborhoods. In a larger context, the abstract -- and the concrete -- structures becoming evident from a characterization of harmony based on mathematics and geometry lead to a profound and more fundamental understanding, similarly to the viewpoint of Rameau who viewed music as another scientific discipline.

We use the following notation for the notes and corresponding integer values of a piano keyboard with its 12 semi-tones and keys:

| d e | g a b | 1 3 | 6 8 10 |

| C D E | F G A B | 0 2 4 | 5 7 9 11 |

The binary tree of notes and implied triads exhibits several relevant local neighborhoods that are discussed next. We consider the tree's top six notes, a sub-tree, arranged in a triangular pattern:

C

e E

g G a

The left triangle covers four 3-note chords:

Cdim = (C,e,g), Cm = (C,e,G), C = (C,E,G),

Caug = (C,E,a). Left (right) tree→move: +3(4).

More generally, this figure captures four 3-note chords, triads, which themselves should be visualized in the plane as (non-degenerate) triangles. The question one must therefore answer is the question concerning a "proper projection." In the figure shown here, two triads define lines, i.e., their projections have become degenerate tringle projections: Cdim shows up as left line/edge C-e-g, and Caug shows up as degenerate line C-E-a. The triads Cm and C can be recognized as triangle projections as their corresponding points in

the plane are not collinear. Their associated (oriented) triangles (simplices) are C-E-G and C-e-G – defining the vertical edge C-G. Thus, one is interested in the existence of an arrangement of these four triads as some kind of note graph (note complex) in three-dimensional space that, when projected into the plane, produces projections showing the four triads as non-degenerate triangles.

It is also relevant to mention that the 3-note **suspended chords** -- Csus2 = (C,D,G) and Csus4 = (C,F,G) – **do not show up** in the local C-neighborhood, at least not when considering this specific tree representation of notes using only minor/major 3rd and 5th steps.

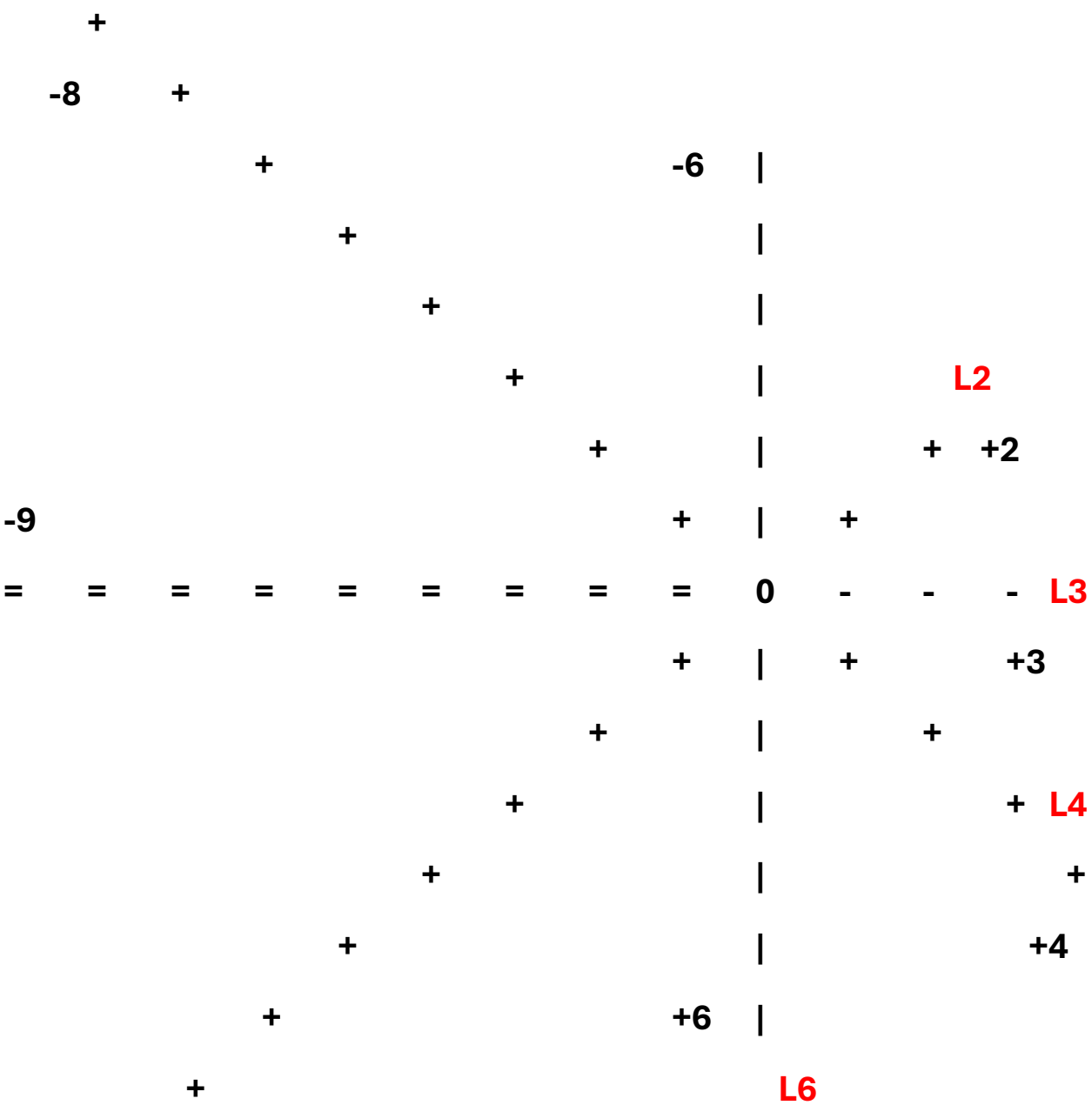
Note. All four triads, in fact all chords, are simply **points on a line**, a discretized one-dimensional axis with integer coordinates, a one-dimensional "path." By defining m3 = minor 3rd and M3 = major 3rd and identifying notes with their corresponding integer values, one obtains: Cdim = C+(m3+m3)), Cm = C+(m3+M3)), C = C+(M3+m3)) and Caug = C+(M3+M3)). In other words, the chord's integer value sequences on the one-dimensional line are Cdim = 0, 3, 6; Cm = 0, 3, 7; C = 0, 4, 7, and Caug = 0, 4, 8. Thus, in principle, there are "no triangles to project" – only one-dimensional paths, line segments or possibly "twisted polylines" in 3D space should possibly be considered for a meaningful projection.

Therefore, it might be more appropriate to think of a 3-note chord, especially a triad (or a 4-note chord, especially a 7th chord) as a "twisted polyline" in 3D (or higher-dimensional?) space. Thus, one would understand a chord as a closed, periodic, cyclic polygon or polyline in space – being closed, cyclic since a chord's root note maps to the start and the end vertex of such a polygonal chord

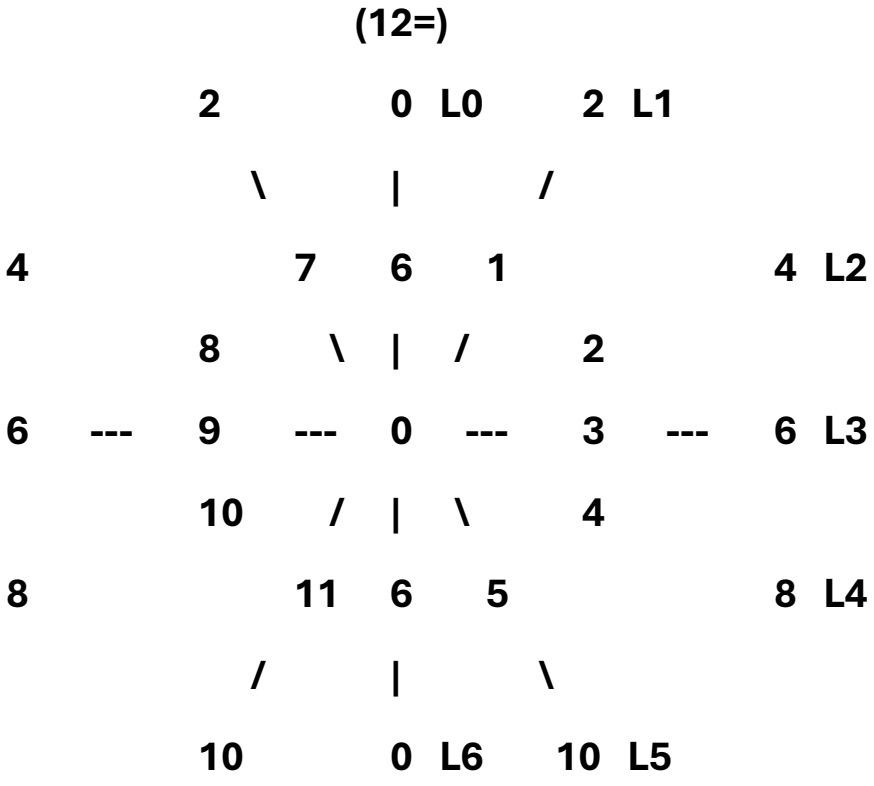
representation. For example, C = (C,E,G) could be mapped to the four vertices v1=C, v2=E, v3=G, v4=v1=C of some graph.

Yet another way of thinking about line directions potentially useful for mapping notes to/onto is based on a generalization of the three line directions (for m3rds, M3rds and 5ths for scale-upward tone movements) occurring in the Tonnetz: One can consider (in the plane) six lines implying 12 line directions, where each line defines upward '+' and downward '-' movements via positive or negative incremental values of mutually different values: The six lines and their associated 12 directional increments can be defined for +1(-11) semi-tone, +2(-10) semi-tone, +3(-9) semi-tone, +4(-8) semi-tone, +5(-7) semi-tone, and +6(-6) semi-tone movements. One can think of this analogy: "The six lines pass through the center of a clock indicating 12 hours on a circle around this center." Similarly to the three directed lines of the standard Tonnetz, one could now use the six lines and their associated 12 directions to define a tessellation and map notes to vertices and chords to tiles for all possible upward or downward movements along the six lines.

The following figure is a simple illustration of a four-line directional arrangement, where four lines pass through the origin '0' and indicate positive-direction movements by increments of +2, +3, +4 and +6 (semi-tones) and negative-direction movements by decrements of -10, -9, -8 and -6, respectively. Since modulo-12 arithmetic is employed, a pair of corresponding incremental and decremental values adds up to 12 – and 12 mod 12 = 0.



Four of the six lines are shown: L2, L3, L4, L6. On these lines, one moves by the positive or negative values as shown. For example, if '0' stand for C, then the positions indicated as +2, +3 and +4 will refer to the notes D, Eb and E, respectively.



This figure shows six distinct lines passing through the “center 0”:

- L0: defined by pair (0, 12=0]) → octaves
- L1: defined by pair (1, 11) → circle of m2nd s and M7th s
- L2: defined by pair (2,10) → circle of M2nds and m7ths
- L3: defined by pair (3, 9) → circle of m3rds and M6ths
- L4: defined by pair (4, 8) → circle of M3rds and m6ths
- L5: defined by pair (5, 7) → circle of 4ths and M5ths
- L6: defined by pair (6, 6) → circle of m 5ths (tritone)

Note. The two integers of a pair add up to 12, being12 mod 12 = 0 in modulo-12 arithmetic. Conceptually, one can also interpret the implied six distinct lines as 12 “outward-directed rays,” emanating from the “center 0” with directions pointing to hour marks for 12 hours on a 12-hour clock, placed uniformly on a circle with “center 0.” Nevertheless, due to the considered modulo-12 setting and the

“duality condition” that the integers of a pair add up to 12, the sequence of clockwise hour directions we use are 0(=12), 1, 2, 3, 5, 5, 6 and 12(=0), 11, 10, 9, 8, 7, 6. In other words, we use “incremental values of +1 on the right side of the clock” and “incremental values of -1 on the left side of the clock.”

One can view the “center 0” geometrically as north pole of a sphere and interpret each of the six distinct lines as great circles on the sphere passing through the north and south poles. Further, since the poles define an axis, the angles between two great circles next to each other should be 360 degrees / 12 = 30 degrees. One can map the notes belonging to each of these “great circles of constant note intervals” onto the corresponding great circles on the sphere.

For example, when using the note C as center 0 and north pole, the note sequences mapped onto the great circles are the following:

- L0: 0 (unison, north pole being a singular/degenerate point)
- L1: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 0 =
  C, d, D, e, E, F, g, G, a, A, b, B, C (12-tone chromatic scale)
- L2: 0, 2, 4, 6, 8, 10, 0 = C, D, E, g, a, b, C
- L3: 0, 3, 6, 9, 0 = C, e, g, A, C (fully diminished 7^{th} chord)
- L4: 0, 4, 8, 0 = C, E, a, C (augmented triad)
- L5: 0, 5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7, 0 =
  C, F, b, e, a, d, g, B, E, A, D, G, C (circles of 4^{th}s and 5^{th}s)
- L6: 0, 6, 0 = C, g, C (tritone)

These sequences are implied by integer-arithmetic facts: 12/1 = 12 (L1); 12/2 = 6 (L2); 12/3 = 4 (L3); 12/4 = 3 (L4); 60/5 = 12 [12 4^{th}s from C to C] (L5); 12/6 = 2 (L6). “Dual lines” are called L6’, L5’, L4’, L3’, L2’ and L1’, respectively. These “dual lines” in opposite direction are

defined by these facts: 12/6 = 2 (L6'); 84/7 = 12 [12 5ths from C to C] (L5'); 24/8 = 3 (L4'); 36/9 = 4 (L3'); 60/10 = 6 (L2'); and 132/11 = 12 (L1'). Also, one can define the lines L0 and L0', implied by "12/0 = 0" [by definition] (L0) and 12/12 = 1 [octave from C to next C] (L0').

Prime number factorization can be used as well in this context:

- 12 = 2*2*3 (12 semi-tones defining an octave)
- The prime number product 2*2*3 captures these values: 0 = 12 (by definition), 1 (by definition), 2, 3, 4, 6, and 12 =0.
- The numbers 7, 8, 9, 10 and 11 have factors in their prime number factorizations that are not in the product 2*2*3. The factors that are not in 2*2*3 are the additionally required factors 7, 2, 3, 5 and 11, respectively – since 7 is 1*7. 8 = 2*2*2, 9 = 3*3, 10 = 2*5, and 11 = 1*11. Thus, one must multiply 12 = 2*2*3 by the factors 7, 2, 3, 5, and 11, respectively, to obtain all needed factorizations and products: 2*2*3*7 = 84, 2*2*3*3 = 36, 2*2*3*5 = 60, and 2*2*3*11 = 132, respectively.

Thus, division and prime number factorization are the operations that define the lengths of these integer/note sequences. One can visualize these circles by using the numbers of octaves needed to start and end with 0 (on C). The numbers of octaves required to perform a "cyclic journey from a start C to an end C" -- i.e., the numbers (2*2*3)*1 = 1, (2*2*3)*5 = 60, (2*2*3)*7 = 84, (2*2*3)*2 = 24, (2*2*3)*3 = 36, and (2*2*3)*11 = 132 -- are the lengths of the octave sequences in the following list, where the 12 semi-tones of the chromatic scale are 0, 1, 2, 3, 4, 5, 6, 7, 8 ,9, t (ten), and e (eleven):

L0: 0 (root note 0 only, unison)

L1: 0123456789te0 (one octave)

L2: 0123456789te0 (one octave)

L3: 0123456789te0 (one octave)

L4: 0123456789te0 (one octave)

L5: 0123456789te0123456789te0123456789te

0123456789te0123456789te0 (five octaves, circle of 12 4ths)

L6: 0123456789te0 (one octave, tritone)

L0': 0123456789te0 (one octave)

L1': 0123456789te0123456789te0123456789te0123456789te

0123456789te0123456789te0123456789te0123456789te

0123456789te0123456789te0123456789te0 (11 octaves)

L2': 0123456789te0123456789te0123456789te

0123456789te0123456789te0 (five octaves)

L3': 0123456789te0123456789te0123456789te0 (three octaves)

L4': 0123456789te0123456789te0 (two octaves)

L5': 0123456789te0123456789te0123456789te0123456789te

0123456789te0123456789te0123456789te0 (seven octaves,

circle of 12 5ths)

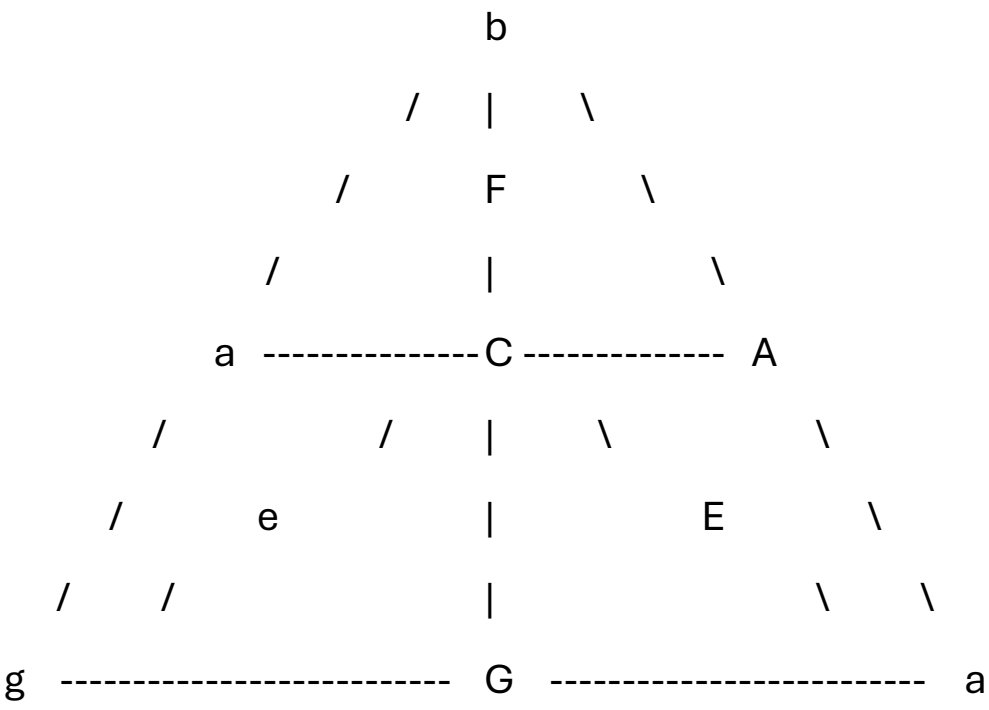
L6': 0123456789te0 (one octave, tritone, identical to L6)

When reducing and compressing the set of octave sequences to the essential integers/notes underlying individual steps, one obtains:

L0:	0	L0':	0-0
L1:	0-1-2-3-4-5-6-7-8-9-t-e-0	L1':	0-e-t-9-8-7-6-5-4-3-2-1-0
L2:	0-2-4-6-8-t-0	L2':	0-t-8-6-4-2-0
L3:	0-3-6-9-0	L3':	0-9-6-3-0
L4:	0-4-8-0	L4':	0-8-4-0
L5:	0-5-t-3-8-1-6-e-4-9-2-7-0	L5':	0-7-2-9-4-e-6-1-8-3-t-5-0
L6:	0-6-0	L6':	0-6-0

Here, each line provides a sequences pair, to emphasize their dual nature, defining the same sequences in reversed orders. One must determine whether this characterization using integer arithmetic, and the implied visual, geometrical representation, captures harmonic relationships of relevance, especially whether one can "see" smooth harmonic progressions.

While the traditional Tonnetz represents harmonically most relevant intervals via its 3-directional line net, one can also devise nets using more than just a 3-directional net. For example, we consider a net that emanates from a center location (e.g., the note C or the integer value 0), and uses six semi-infinite ray directions at this point:



This primitive sketch shows the m 3rd direction C→e→g, the major 3rd direction C→E→A and the perfect 4th direction C→F→b. In addition, there are the directions of the major 5th C→G, the major 6th C→A and the minor 6th C→a. The center vertex (C=0) has rank (valence) six, and its neighbor vertices in counterclockwise order are e, G, E, A, F and a. The center C and its six neighbor vertices define the triads C min = (C,e,G), C maj = (C,E,G), A min = (A,C,E), F maj = (F,A,C), F min = (F,a,C) and Ab maj = (a,C,e). The simple sketch still shows that the center C is surrounded by six "quadrilaterals," i.e., closed polygonal tiles with four vertices. They can be split into two triangles by introducing the edges eG, GE, EA, AF, Fa and ae.

Note. One can also interpret this representation as a projection of a tetrahedron into the plane: The tetrahedron has a "top vertex" C and "bottom-face vertices" g, a and b. The three "side-edges" of the tetrahedron are the edges C-e-g, C-E-a, and C-F-b. The three "bottom-face edges" of the tetrahedron are g-G-a, a-A-b, and b-a-g.

This interpretation is another way of understanding this note graph and “seeing” triads of interest and harmonic relationships in it.

Note. An essential goal of such graph representations is the goal of trying to encode – also geometrically – those semi-tone interval lengths that are harmonically relevant, ideally all 12 lengths, i.e., 0, …, 11. Due to complementarity, it suffices to attempt to represent only half of the 12 semi-tone interval lengths since two interval lengths l1 and l2 that are complementary must satisfy l1+l2 = 12, where l1 and l2 define absolute values, i.e., l2 = 12-l1, one length defining the other. One can illustrate this on a line with underlying modulo-12 arithmetic as follows:

6 7 8 9 10 11 12=0 1 2 3 4 5 6

g G a A b B C d D e E F g

CC

B ---------- d

b ---------- D

A ---------- e

a ---------- E

G ---------- F

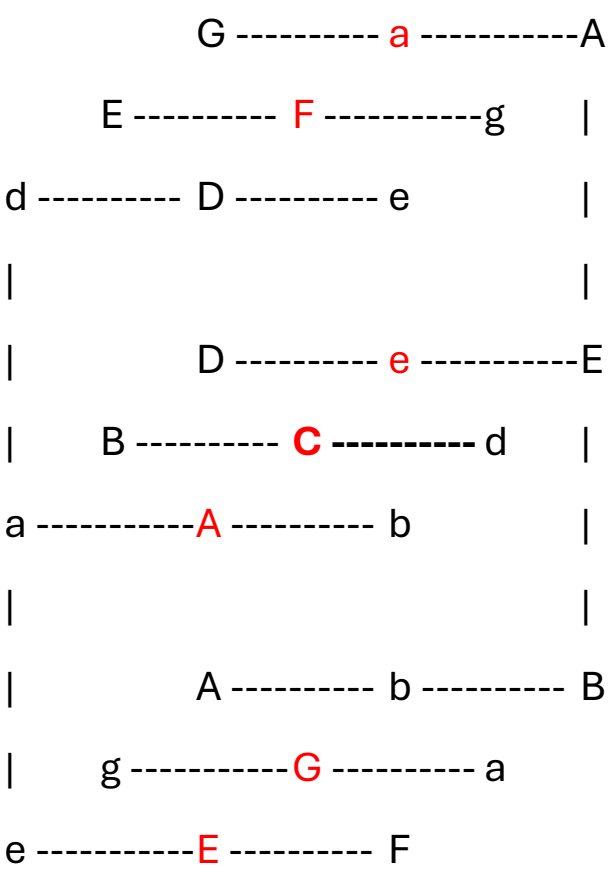
g ---------- g

The center note is C=0; the six semi-tones directly above C are to the right of C; and the six semi-tones directly below C are to the left of C. All shown note pairs have index value sums of 12. Ideally, a geometrical representation of a note graph should include the

complementary length pairs (l1, l2) with values (0,0), (1,11), (2,10), (3,9), (4,8), (5,7), and (6,6), associated with edges in the graph.

Thus, an edge with associated length pair (l1, l2) stands for moving by l1 semi-tone steps in one edge direction and l2 semi-tone steps in the other, complementary, edge direction. One can potentially ignore the (0,0) and (6,6) pairs, as edges with these value pairs might not be relevant for certain harmonic considerations, since one is mainly driven by the identification of smooth progressions. Thus, a graph layout that captures the five value pairs (1,11), (2,10), (3,9), (4,8), and (5,7) should generally be sufficient.

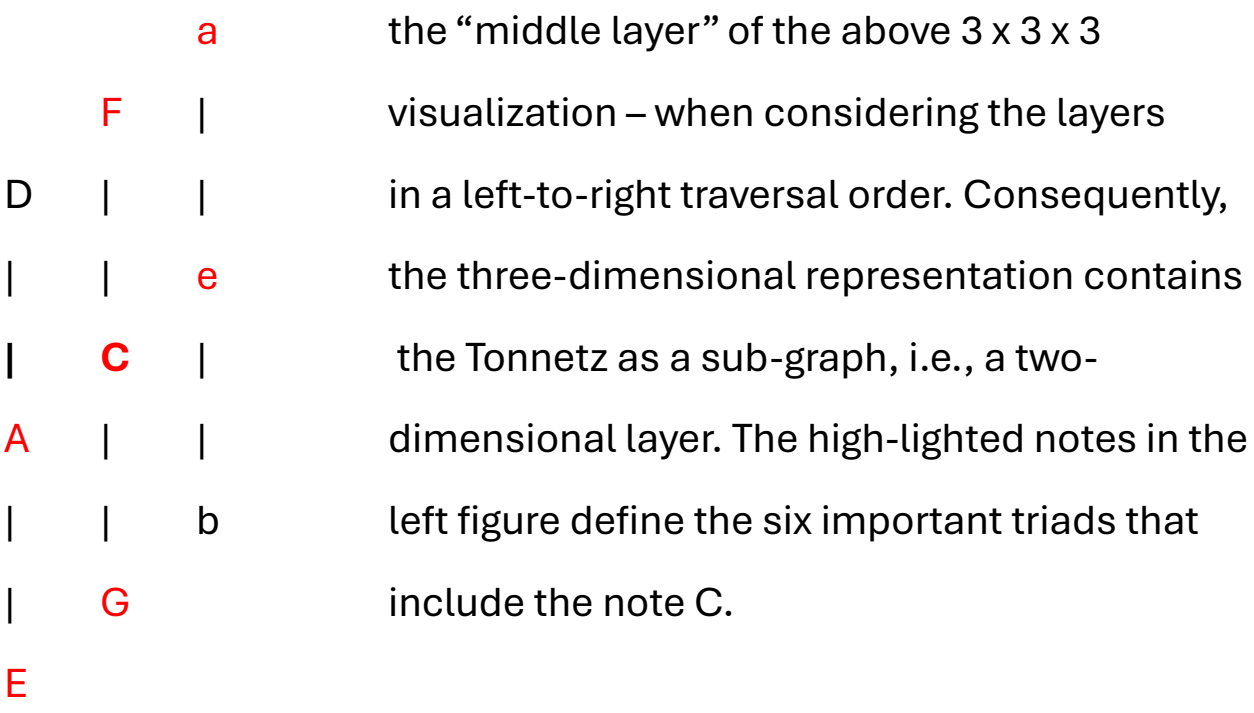
For example, it is possible to associate notes with the “three layers” of a 3 x 3 x 3 rectangular grid. The following figure shows the layers from top to bottom. The three directions are left-to-right, diagonally front-to-back, and bottom-to-top, with assigned note interval pairs.



The pairs are (1,11), (3,9) and (5,7). The center note is C, with B and d as left and right, A and e as front and back, and G and F as bottom and top neighbors. Notes change by a single semi-tone when moving in the rows. All lines from front to back define diminished triads, e.g., the diminished chord A dim = (A,C,e). Moving vertically involves 4ths (upward) and 5ths (downward), e.g., one sees the circle-of-fifth sub-sequence F-C-G. Further, one can also identify relevant chords in the C-neighborhood, including the Cmaj = (C,E,G), Cmin = (C,e,G), Fmaj = (F,A,C), Fmin = (F,a,C), Abmaj = (a,C,e) and Amin = (A,C,E). The triad-relevant notes are high-lighted,

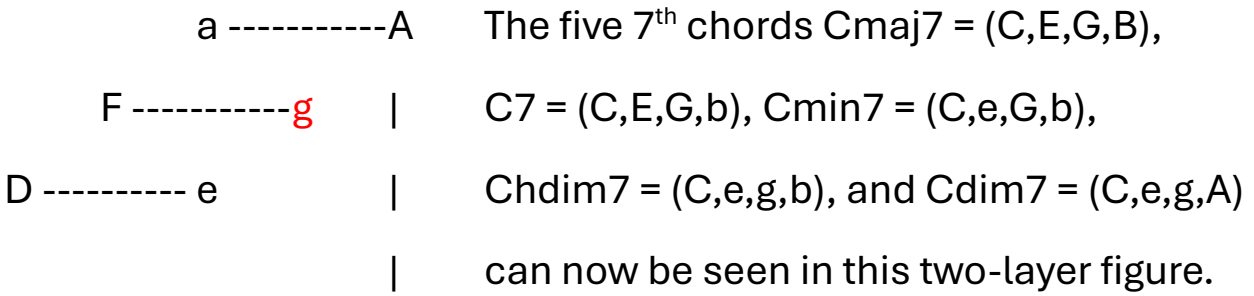
Since this visualization is a “projection” of a note graph that has been placed geometrically in three-dimensional space, one must get used to “seeing” the chords of relevance in this initially unfamiliar and more complicated representation. With some practice, one can locate the mentioned six important C-triads as “triangles” in the shown configuration by searching around C. The

following figure focuses on only that portion of the C-neighborhood that defines these six triads. In fact, these six triads are included in

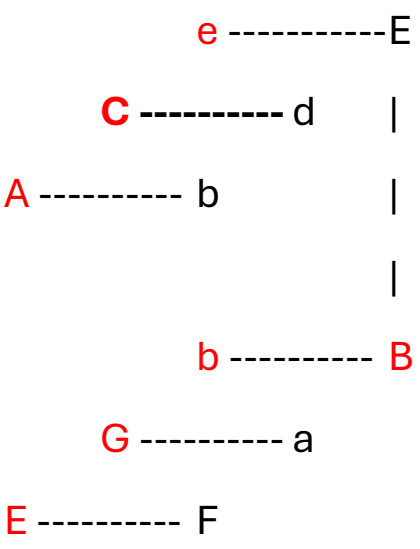


the "middle layer" of the above 3 x 3 x 3 visualization – when considering the layers in a left-to-right traversal order. Consequently, the three-dimensional representation contains the Tonnetz as a sub-graph, i.e., a two-dimensional layer. The high-lighted notes in the left figure define the six important triads that include the note C.

Tow notes that are part of the common C 7th chords are not in this "middle layer": B and g. Both notes are in the layer to the right and to the left of this "middle Layer." By choosing the neighbor layer to the right, one obtains this graph excerpt:

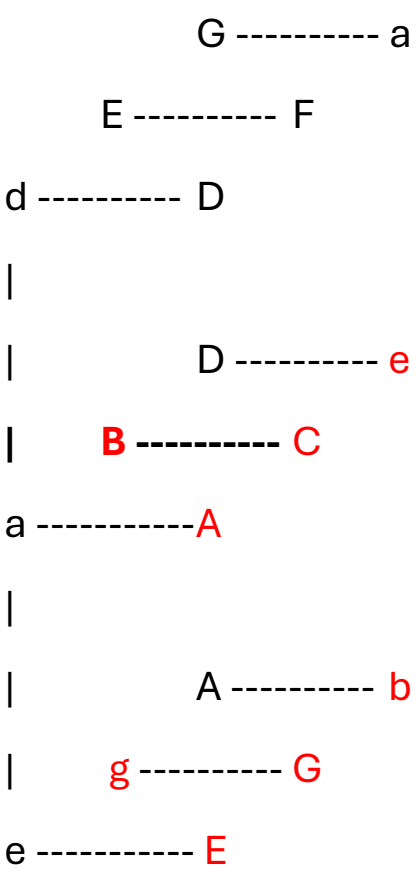


The five 7th chords Cmaj7 = (C,E,G,B), C7 = (C,E,G,b), Cmin7 = (C,e,G,b), Chdim7 = (C,e,g,b), and Cdim7 = (C,e,g,A) can now be seen in this two-layer figure.

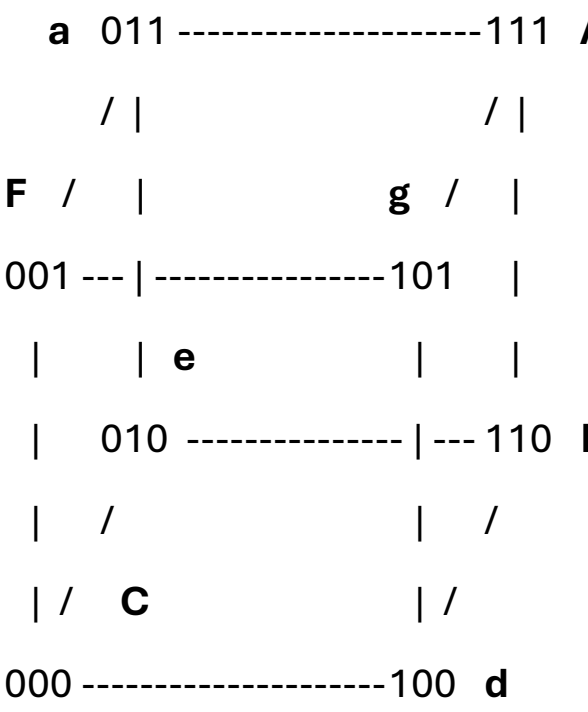


The note sequences of these 7th chords define polylines/polygons of length four, connecting the four notes required for each chord. Some of these polygons lie in a plane in three-dimensional space. Due to the harmonic complexity of these 7th chords, it is “not easy to see them.”

By choosing the left neighbor layer, one obtains the following graph excerpt:



The same five 7th chords Cmaj7, C7, Cmin7, Chdim7, and Cdim7 can also be seen in this tw-layer figure. The notes included in these chords are high-lighted. The note sequences of these 7th chords define polylines/polygons of length four, connecting the four notes required for each chord. Some of these polygons lie in a plane in three-dimensional space. Due to the harmonic complexity of these 7th chords, it is “not easy to see them.”



The left figure focuses on the cube that serves as a building block (three-dimensional tile) of the tessellation discussed. The "natural" directions of the cube are left-right, front-back, and bottom-top directions. These directions are shown in the figure as line segments.

In addition, one can conceptually think of, represent and visualize additional directions that one can associate with the cube – and use them for the ordering/indexing of notes in a note graph. First, each of the six quadrilateral faces of the cube can be split by using the face diagonals as split edges. The cube vertices induce 12 such edges:

Bottom, top faces:	(000, 110), (100, 010), (001, 111), (101, 011)
	(interval pairs: (4, 8), (2, 10), (4, 8), (2, 10))
Front, back faces:	(000, 101), (100, 001), (010, 111), (110, 011)
	(interval pairs: (6, 6), (4, 8), (6, 6), (4, 8))
Left, right faces:	(000, 011), (010, 001), (100, 111), (110, 101)
	(interval pairs: (8, 4), (2, 10), (8, 4), (2, 10))

Second, the cube has four interior diagonals, connecting vertices that are opposite to each other. The induced four edges are:

Interior edges: (000, 111), (100, 011), (010, 101), (110. 001)

(interval pairs: (9, 3), (7, 5), (3, 9), (1, 11))

When using these additional edges for moving from note to note, from vertex to vertex, each edge will again have two interval length values associated with it, for moving upward and moving downward. These interval lengths pairs, with lengths adding up to 12, are shown above, directly below the triple-index pairs that define the edges. It will depend on the specific questions one has about harmonic relationships to decide which of all these possible edges and their associated note interval lengths values are most appropriate for finding answers and gaining insights.

Note. Since the number 12 is the number of semi-tones defining the chromatic scale and an octave music theoreticians explore whether the icosahedron (with 12 vertices, 30 edges, 20 faces) and the dodecahedron (with 20 vertices, 30 edges, 12 faces) provide useful nets/networks, as they define “closed, cyclic tessellations of the sphere.” These Platonic solids – solids, NOT surfaces -- are dual to each other, since the vertices and faces of their associated tessellations dualize. There could potentially exist ways to map notes and chords to the vertices and faces of these periodic tessellations that lead to new insights into progressions and their innovative creation. An objective would be to place notes and chords in such a way that known and unknown, possibly interesting, chord progressions become paths in tile neighborhoods of these

tessellations. A brute-force approach would evaluate the 12! = 479,001,600 combinatorial possibilities.

As it seems prohibitive to generate all 12! possibilities for assigning 12 semi-tones to 12 polyhedral vertices (or faces) and assess their quality for gaining a deeper or better understanding of harmonic relationships, one could consider heuristics to reduce the number of possibilities being explored. For example, one could use the absolute semi-tone distance values between two semi-tones to determine whether one should assign two semi-tones to vertices (or faces) being close to or far away from each other in the polyhedral tessellation. One can catalog the absolute semi-tone distance values, between 0 and 6, in a 12 x 12 cyclic, symmetric matrix:

	C	d	D	e	E	F	g	G	a	A	b	B
C	0	1	2	3	4	5	6	5	4	3	2	1
d	1	0	1	2	3	4	5	6	5	4	3	2
D	2	1	0	1	2	3	4	5	6	5	4	3
e	3	2	1	0	1	2	3	4	5	6	5	4
E	4	3	2	1	0	1	2	3	4	5	6	5
F	5	4	3	2	1	0	1	2	3	4	5	6
g	6	5	4	3	2	1	0	1	2	3	4	5
G	5	6	5	4	3	2	1	0	1	2	3	4
a	4	5	6	5	4	3	2	1	0	1	2	3
A	3	4	5	6	5	4	3	2	1	0	1	2
b	2	3	4	5	6	5	4	3	2	1	0	1
B	1	2	3	4	5	6	5	4	3	2	1	0

The smallest values 0 and 1 and the maximal value 6 are in matrix diagonals. These values could be considered for note placement.

There are six semi-tone pairs with notes having maximal distance value 6: (C,g), (d,G), (D, a), (e,A), (E,b) and (F,B). Thus, one could consider placing the two notes defining a pair such that the two notes also have maximal distance on the Platonic solid used to map them to; or, alternatively, one could place them such that the notes of such a pair with maximal note distance have minimal/small distance on the Platonic solid. For example, since the number of these pairs is six, one could consider mapping the notes of each note-pair to opposite vertices (of an icosahedron) or opposite faces (of a dodecahedron.

The numbers of original vertices, edges and faces of these Platonic solids might not be sufficient to encode all the harmonic elements one wants to map to the associated elements of the tessellations. There could be too many notes and chords to be represented. Thus, one could utilize “subdivision techniques” known in geometric modeling, for example. Such techniques split a triangular face of an icosahedron into four sub-triangles (by introducing a vertex in the center of an original triangle), or they subdivide a pentagonal face of a dodecahedron into five sub-triangles (by introducing a vertex in the center of an original pentagon). One can connect the added vertex by connecting it to the vertices of the original triangular (pentagonal) face it has been inserted into.

Further, one could also associate notes and chords with edges in these tessellations, and one could use subdivision techniques that insert additional vertices in the interior of original faces as we well

along original edges. It is noteworthy to recognize that the resulting refined, higher-resolution tessellations produced by subdivision techniques define potentially useful “geometrical hierarchies.”

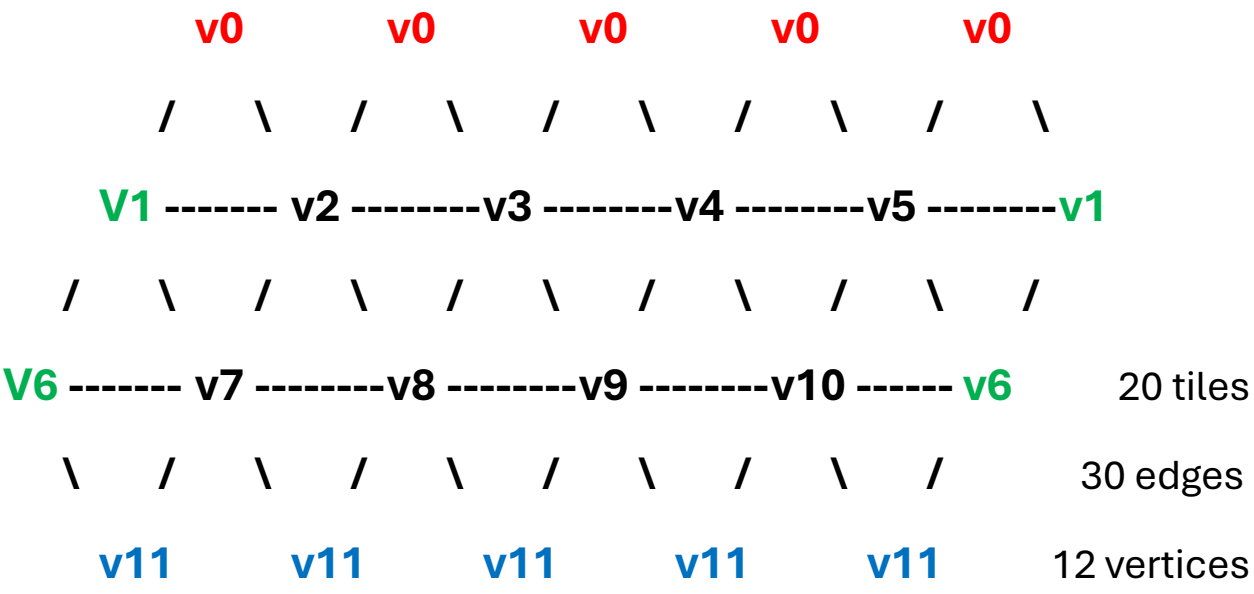
It is important to remind oneself what a main purpose is when one devises ways to represent and visualize notes, chords and scales:

- **The objective is to construct a representation that places notes, chords and scales that are “close” to each other, i.e., that have “small distances” to each other in a specific and harmonically meaningful way, in close visual proximity. In other words, the “building blocks” of harmony (notes and chords) that are close to each other should be mapped to “representational primitives” that are also close to each other. A visual representation should be highly effective: It should make it possible to recognize interesting harmonic relationships easily and quickly. In addition, since the local “group” of these harmonic relationships is repetitive -- the “group” pattern repeating for all possible root notes, all keys -- the way notes are placed is crucial. The placement should allow one to see and perform smooth step-by-step, group-to-group, key-to-key movements, i.e., movements that are harmonic progressions leading to only minimal changes to chords, for example.**

Of course, visual representations of harmonic relationships that map notes and chords to Platonic solids in three-dimensional space are hard to visualize, hard to project and hard to comprehend. One can use “unfolded” (or “flattened”/“projected”) tessellations of (the outer surfaces of) these solids in the plane. As a result, one obtains

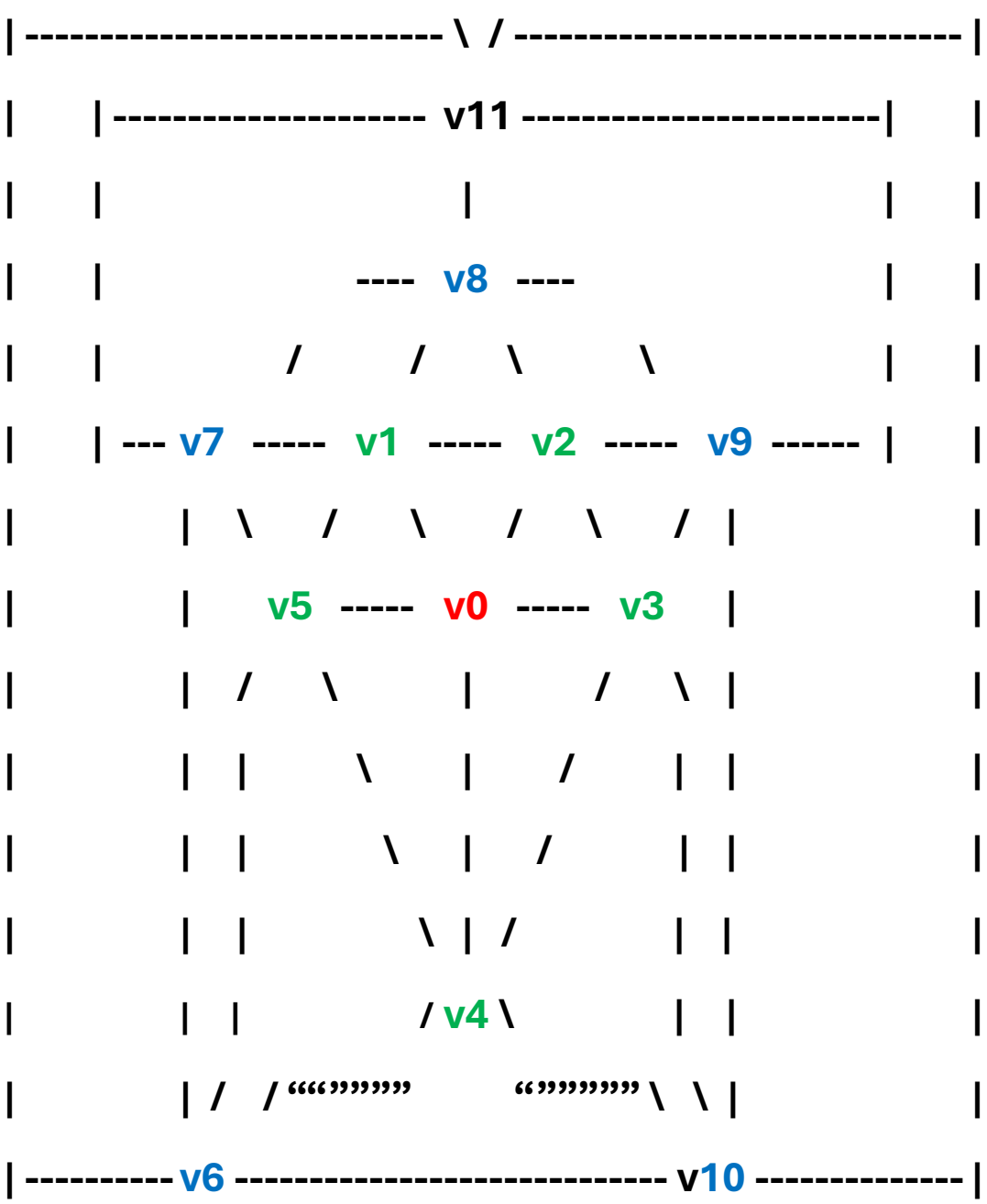
co-planar triangles or pentagons, with connectivity among tiles as defined by the Platonic solids. First, we consider an “unfolded” representation of the icosahedron.

Unfolded icosahedron



This unfolded surface triangulation is “periodic.” It defines a periodic triangle graph that results when unfolding the tessellation of the icosahedron’s associated outer surface. Here, the top vertex is called v0 and the bottom vertex is called v11. The left boundary and the right boundary in the representation reflect the underlying periodicity. There are five top and five bottom triangles, and there are ten side triangles. There exist 30 different edges, but in the representation several of the shared edges are drawn twice (for the top and bottom triangles). Of importance is the fact that the number of vertices is equal to the number of semi-tones of the chromatic scale. Second, we consider a “flattened ”or “projected” representation of the icosahedron.

Flattened / projected icosahedron

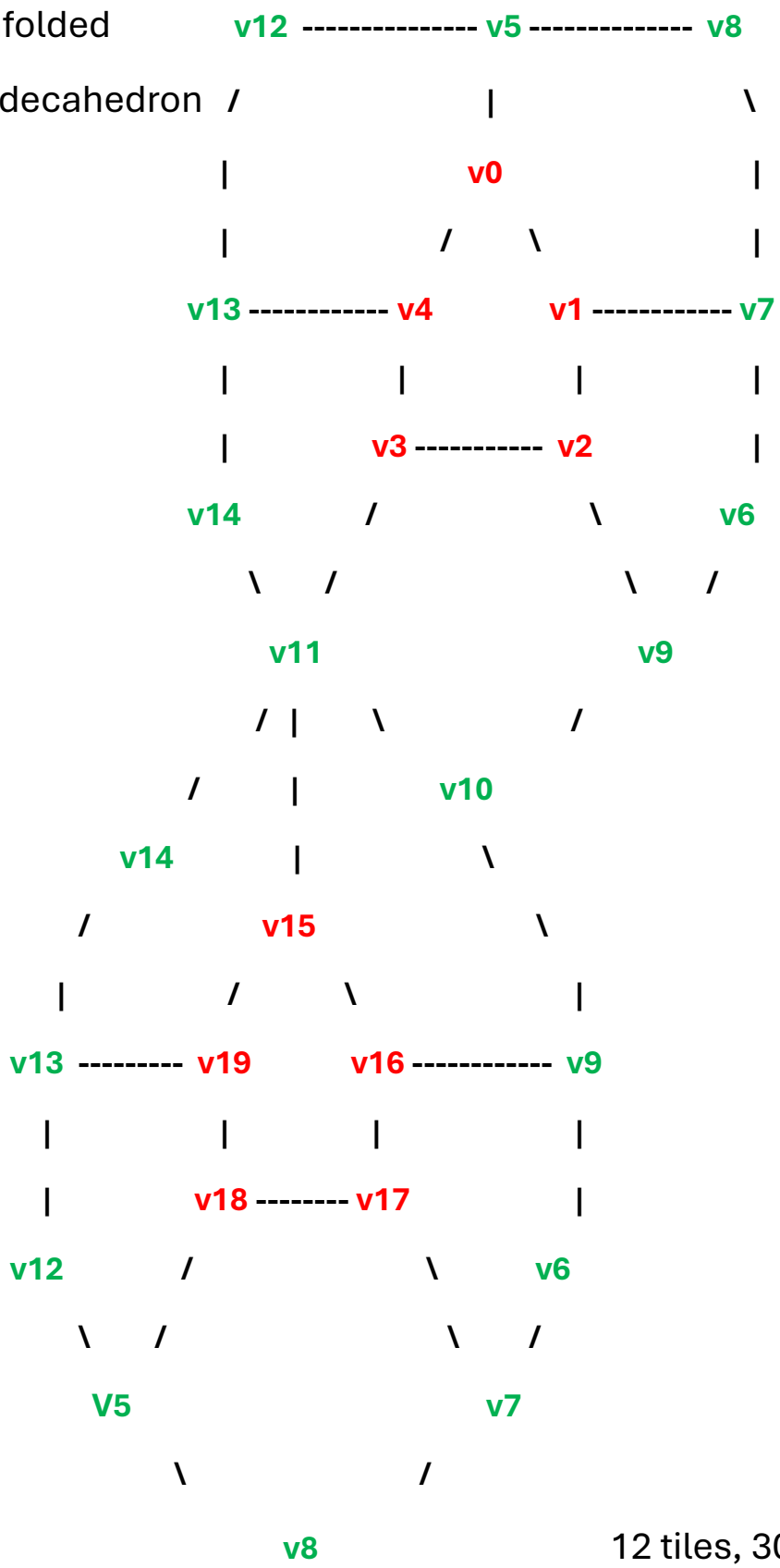


20 tiles, 30 edges, 12 vertices

This simplified graph representation does not contain any crossing edges. Consequently, edges are not straight-line segments, they are simply tile boundaries connecting three vertices. With practice, it is straightforward to “read” such a planar representation. While it seems that there are only 19 tiles, there exists a 20^{th} tile: The 20^{th} tile is the tile with vertices v6, v10, and v11, the “rest of the plane.”

Next, we consider an unfolded and flattened/projected tessellation of the dodecahedron, leading to pentagonal tiles in the plane. Again, the planar tessellation must reflect the fact that the dodecahedron’s outer surface is closed. Two figures are simple sketches of the unfolded and flattened/projected dodecahedron.

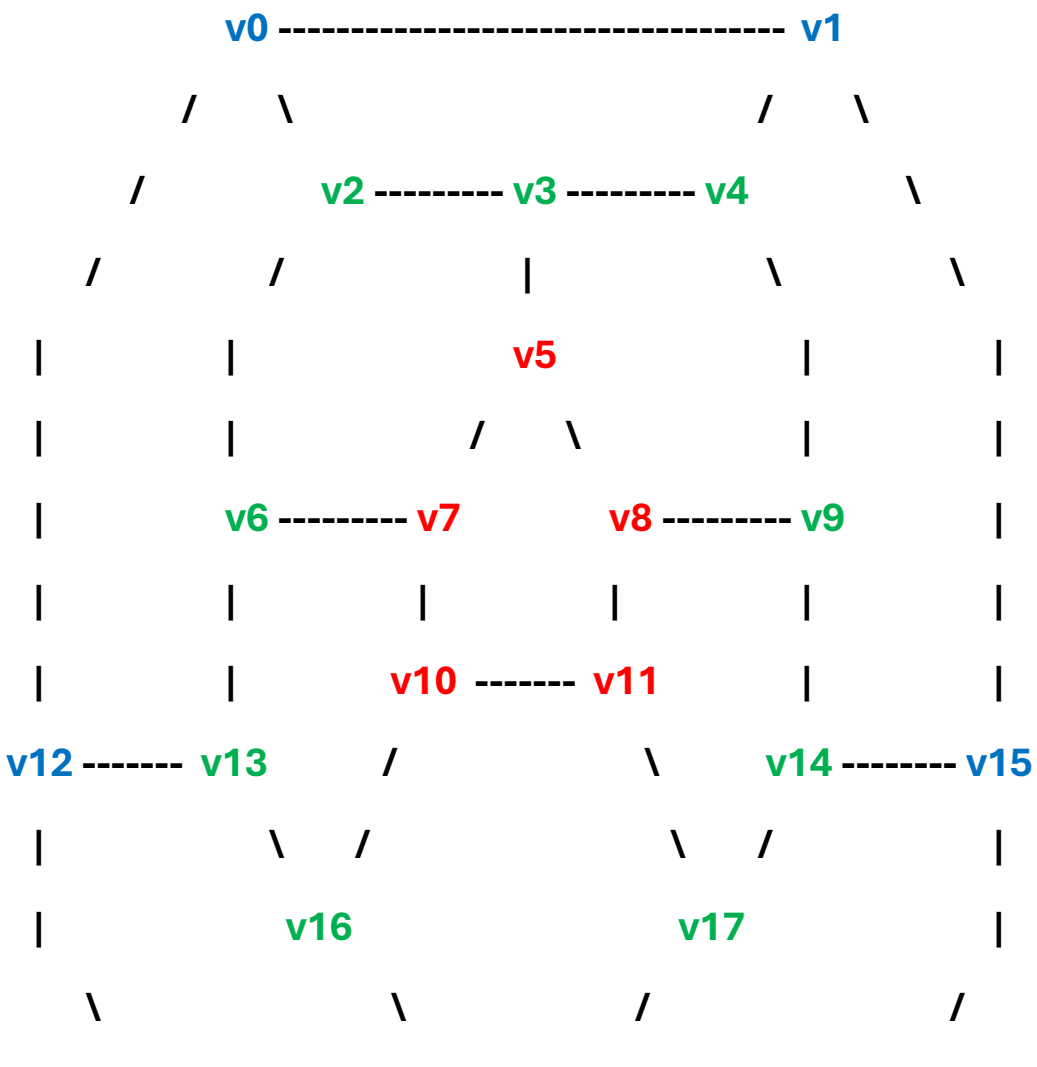
Unfolded dodecahedron



12 tiles, 30 edges, 20 vertices

These rather abstract sketches of possible representations of the graphs in the plane do not properly reflect the actual distances between vertex positions or tile areas of Platonic solid tessellations. They are correct merely in a topological; sense, i.e., they correctly capture the connectivity between vertices and tile neighborhoods.

Flattened / projected dodecahedron



12 tiles

30 edges

20 vertices

While it seems that there are only 11 tiles, there exists a 12^{th} tile: The 12^{th} tile is the tile with vertices v0, v1, v12, v15 and v19. Again, the important fact is the fact that the number of semi-tones of the chromatic scale is equal to the number of tiles of the unfolded dodecahedron, and a relevant question asks whether one can map semi-tones in a meaningful way to the 12 tiles.

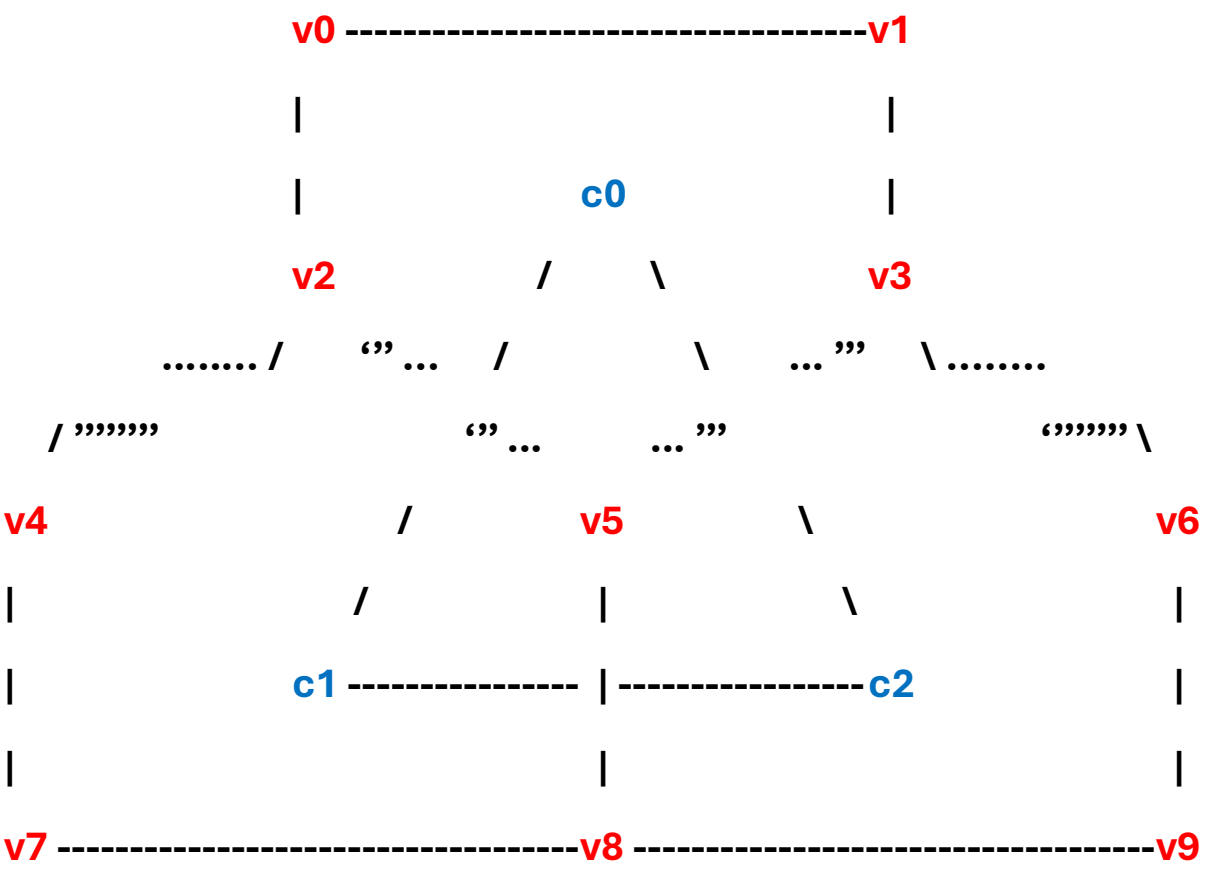
It is routine to define these tessellations via tables listing their respective vertex, edge and tile indices, where vertex indices uniquely define the vertices being the end vertices of edges and the corner vertices of triangular and pentagonal tiles, respectively.

Note. The two triangular and pentagonal tessellations are dual to each other. Each vertex in the tessellation of the dodecahedron has valence (rank) three and corresponds to a triangle in the tessellation of the icosahedron. Three pentagonal tiles always share a common vertex in the tessellation of the dodecahedron, and such sets of pentagonal tiles correspond to the sets of three vertices defining the vertices of the triangles in the tessellation of the icosahedron. Each vertex in the tessellation of the icosahedron has valence (rank) five and corresponds to a pentagon in the tessellation of the dodecahedron. Five triangular tiles always share a common vertex in the tessellation of the icosahedron, and such sets of triangular tiles correspond to the sets of five vertices defining the vertices of the pentagons in the tessellation of the dodecahedron.

Considering these two (planar) tessellations, one can explore their use for mapping notes and chords to their defining elements, i.e., to vertices and tiles, and possibly to their edges. Many possibilities

exist to define a mapping, and one must keep in mind the most important harmonic characteristics one wants to capture in an easily recognizable way when evaluating possible mappings.

One can utilize the duality of the triangle-tessellation of the icosahedron and the pentagon-tessellation of the dodecahedron. For “note distance and chord mappings,” a tessellation should support the encoding of the numbers that define semi-tone interval distances, 1, 2, 3, 4, 5 and 6, and the numbers of notes defining chords 1, 2, 3 and 4. Considering periodicity of the chromatic scale and using modulo-12 arithmetic, semi-tone distances of 7, 8, 9, 10 and 11 do not need to be encoded; chords of interest are 1-, 2-, 3- and 4-note chords. A combined, super-imposed triangular-quadrangular-pentagonal tessellation can support an encoding.



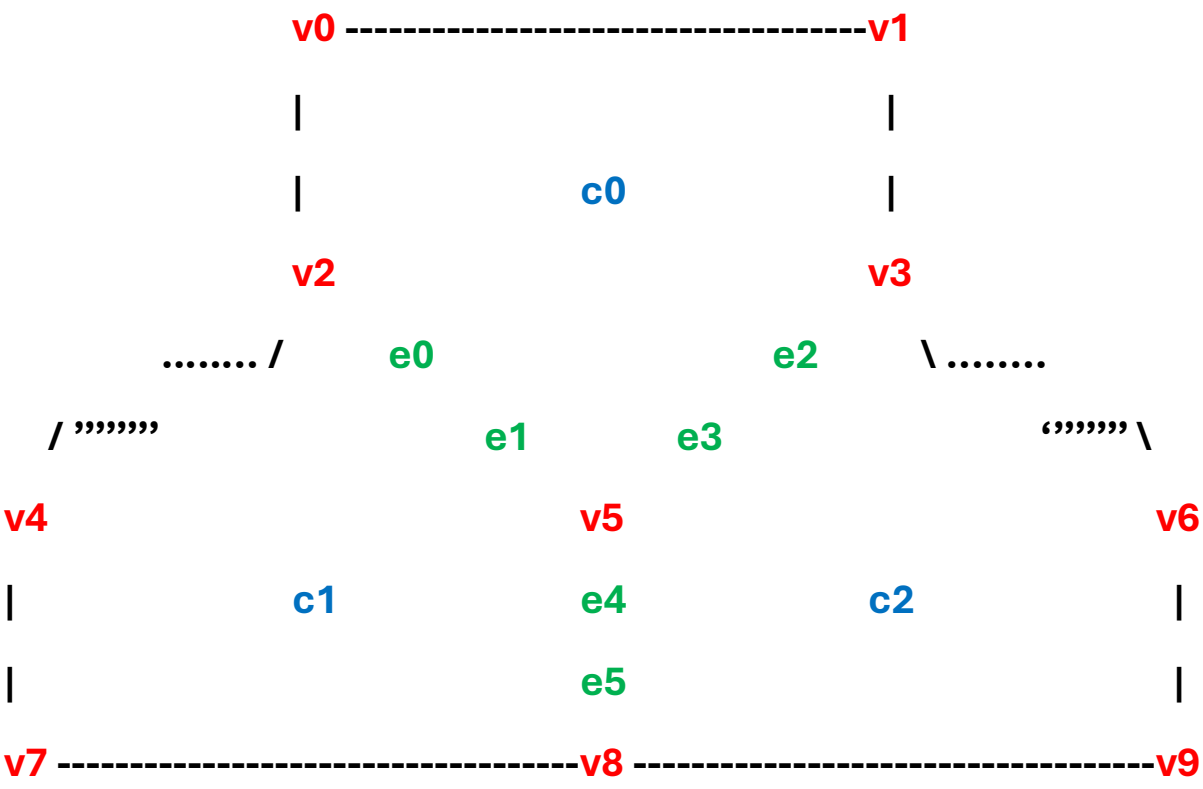
This simple sketch shows three pentagons defined by the vertex sets {v0,v1,v2,v3,v5}, {v2,v4,v5,v7,v8} and {v3,v5,v6,v8,v9}. They share the common vertex v5. One must understand that these three coplanar pentagons are only a part of a complete and topologically closed unfolded or flattened/projected tessellation of the dodecahedron. The figure also includes center vertices for each pentagon, called c0, c1 and c2, and they are used to establish edges between pairs of c-vertices that lie in neighbor pentagons. These edges define triangles that define the "triangle complex" being dual to the "pentagonal complex." The triangle with vertices c0, c1 and c2 in the figure one example. Further, one can subdivide each pentagon into five triangles by connecting each c-vertex with the five vertices of the pentagon containing it. Conceptually, the overall resulting triangulation can be viewed as a superposition of a triangulation onto the set of pentagons. The triangle defined by vertices v0, v1 and c0 is an example. One can also "see" quadrilaterals, each defined by two triangles, which in turn are defined by two c-vertices of neighbor pentagons and two v-vertices defining the edge shared by the two neighbor pentagons. One of those quadrilaterals has the vertices c0, v2, v5 and c1. This is a summary of the construction:

- The v- and c-vertices are 0-dimensional primitives.
- All edges are 1-dimensional primitives.
- The triangles, quadrilaterals and pentagons are 2-dimensional primitives.
- The v-vertices have "initial" valence (rank) 5; inserting the c-vertices and connecting them with their five surrounding v-vertices, the updated valence of v-vertices increases to 6, and c-vertices have valence 2x5 = 10, assuming that c-vertices are

also connected with the five c-vertices of the neighbor pentagons. Depending on context, one can thus use a c-vertex as a vertex of valence 5 or valence 10.

There does not exist a closed tile/polygon with six vertices in this construction, but this is most likely not relevant. Next, one must determine how to map notes and chords of interest to the 0-, 1- and 2-dimensional primitives with 3, 4 and 5 edges.

A more complicated subdivision proves applied to the given co-planar pentagonal tessellation of the dodecahedron is described next. It uses additional e-vertices inserted in the interior of edges that connect v-vertices. The e-vertices are shown in this figure:



This figure again shows three pentagons. The three pentagons are defined by the three vertex sets {v0,v1,v2,v3,v5}, {v2,v4,v5,v7,v8} and {v3,v5,v6,v8,v9}, Center vertices c0, c1 and c2 are inserted into the

pentagons. The shared pentagon edges, defined by vertex pair sets {v2,v5}, {v3,v5} and {v5,v8}, have two e-vertices in their interior, {e0,e1}, {e2,e3} and {e4,e5}, respectively. In addition to the edges defined for the simpler construction above, one can establish edges (not shown) by connecting c0 with e0, e1, e2 and e3; by connecting c1 with e0, e1, e4 and e5; and by connecting c2 with e2, e3, e4 and e5. Thus, one "sees" the implied dual triangle defined by center vertices, c0, c1 and c2, and triangles/quadrilaterals obtained by connecting c-vertices with e-vertices, defined by vertex sets {c0,e0,e1,c1}, {c0,e2,e3,c2} and {c1,e4,e5,c2}. The resulting "subdivided and combined tessellation" consists of three pentagons, one (dual) triangle and three quadrilaterals, which in turn can be viewed as two triangles. In fact, one can "see" three additional quadrilaterals associated with a v-vertex. Considering v-vertex v5, its associated quadrilaterals are implied by the three vertex sets {c0,e1,e3,v5}, {c1,e1,e4,v5} and {c2,e3,e4,v5}. As a consequence, one can now even consider a six-sided tile/polygon defined by the vertex sequence c0, e1, c1, e4, c2, e3 (,c0). This more complicated construction supports the encoding of the numbers 1, 2, 3, 4, 5 and 6 via the primitives available in the tessellation components with varying dimensionality. This is a summary:

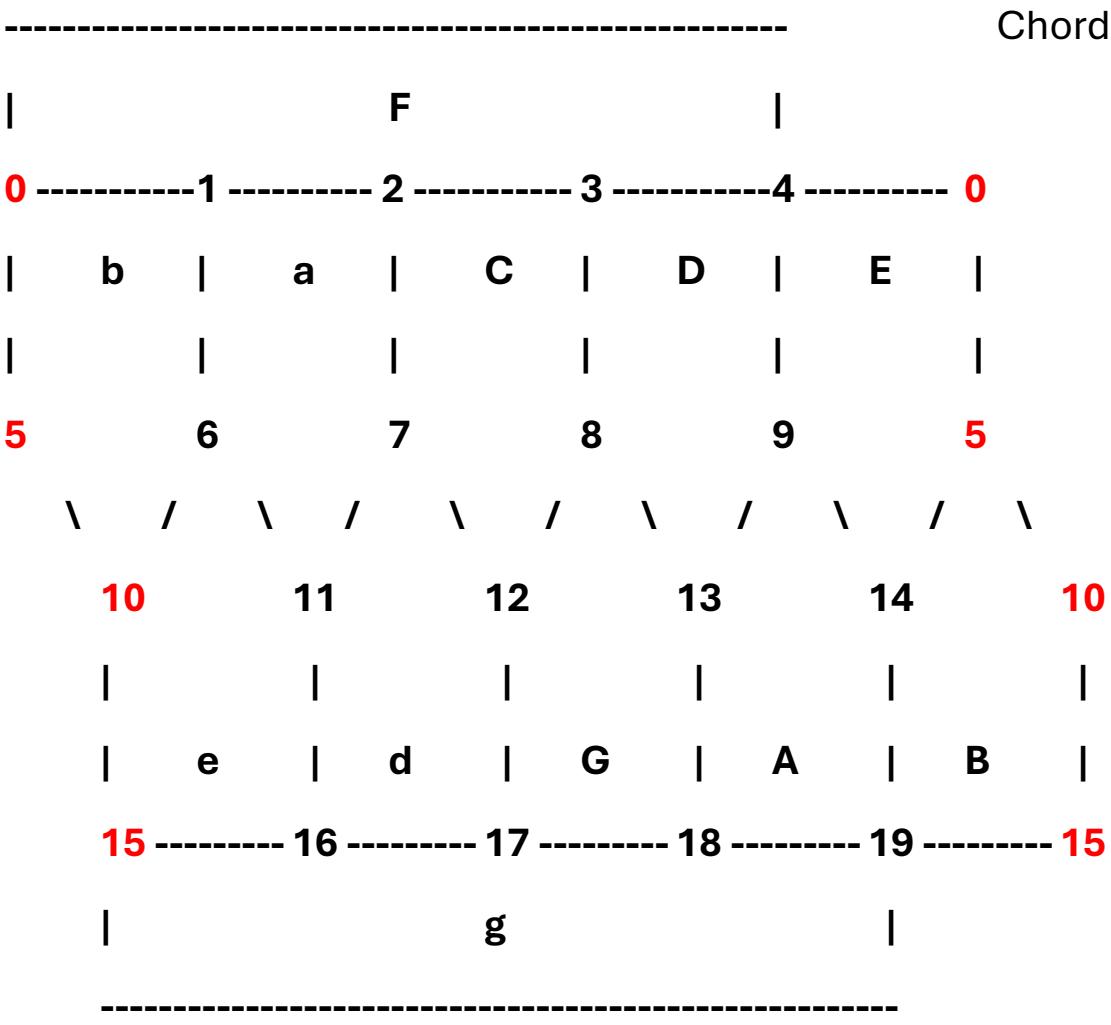
- Each original v-vertex (in fact, every vertex) is a 0-dimensional primitive, understandable as a 1-note chord (unison). Additional fact: Each v-vertex initially has valence (rank) three.
- Each edge connecting v-, c- or e-vertices (in fact, every edge) is a 1-dimensional primitive, understandable as a 2-note chord.
- The triangle defined by the c-vertices is a three-sided primitive, understandable as a 3-note chord. Additional fact: Three c-

vertices share or “surround” one original v-vertex; these c-vertices define a (dual) triangle.

- The quadrilaterals defined by c-vertices of neighbor pentagons and their connections with e-vertices on the shared pentagon edges are four-sided primitives, understandable as 4-note chords. Additional fact: Connecting each c-vertex with two e-vertices on each of its five pentagonal tile boundary edges establishes each c-vertex as a vertex of valence (rank) 2 x 5 = 10. (One can “see” and could discuss even more sub-structures that would impact valence etc.)
- The original pentagons themselves are five-sided primitives, understandable as 5-note chords.
- The six-sided tile defined by the vertex sequence c0, e1, c1, e4, c2, e3 (, c0) is a primitive understandable as a 6-note chord.

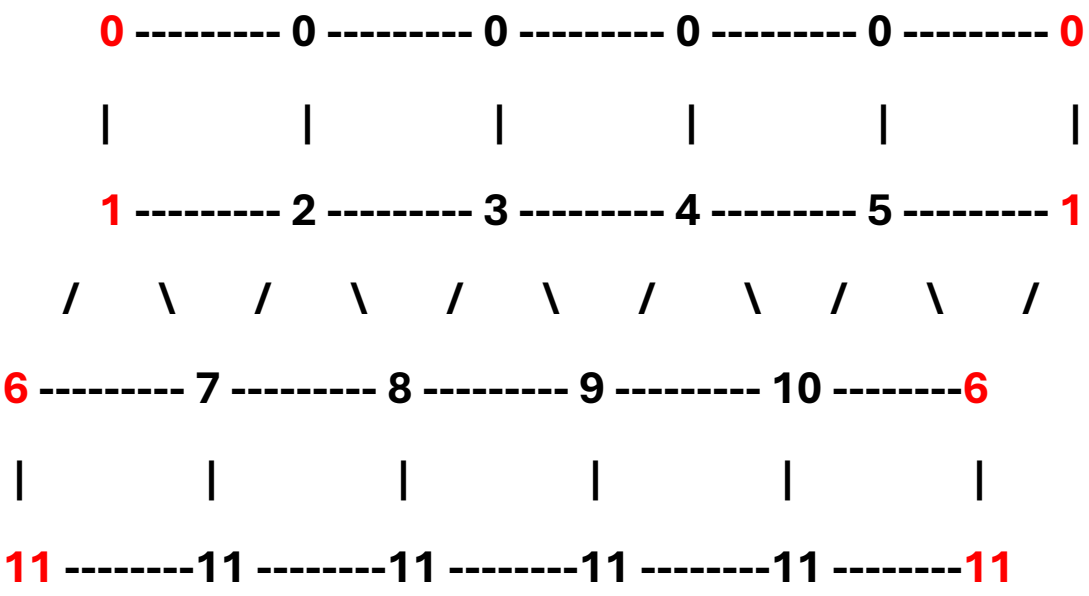
Thus, the created subdivided tessellation, constructed as a combination or a “super-position” of triangles, quadrilaterals, pentagons and hexagons, provides the desired degrees of freedom to map semi-tone notes, semi-tone note distances and chords. While the two figures above focus on the neighborhood of three pentagons only, this local configuration is the building block that can be extended in the plane, keeping the tessellation periodicity of the icosahedron and dodecahedron in mind (or not). The challenge is to find a mapping of notes and chords to vertices, edges and tiles in such a tessellation, ensuring that the mapping serves one’s specific purposes for encoding and recognizing (easily!) the harmonic relationships of interest.

The following sketches illustrate ways to map notes and chords to the described tessellations. **The overarching imperative "placing notes and chords that are harmonically close to tessellation primitives that are close."** A few examples are considered in the following. The next figure shows a planar graph representation of the dodecahedron and its 12 tiles. One can think of this representation as an "unrolled version" of the dodecahedron, with the left and right boundary polygons of the sketch (having the same vertices) defining circular periodicity, while the top and bottom pentagons represent the top and bottom tiles of the dodecahedron.



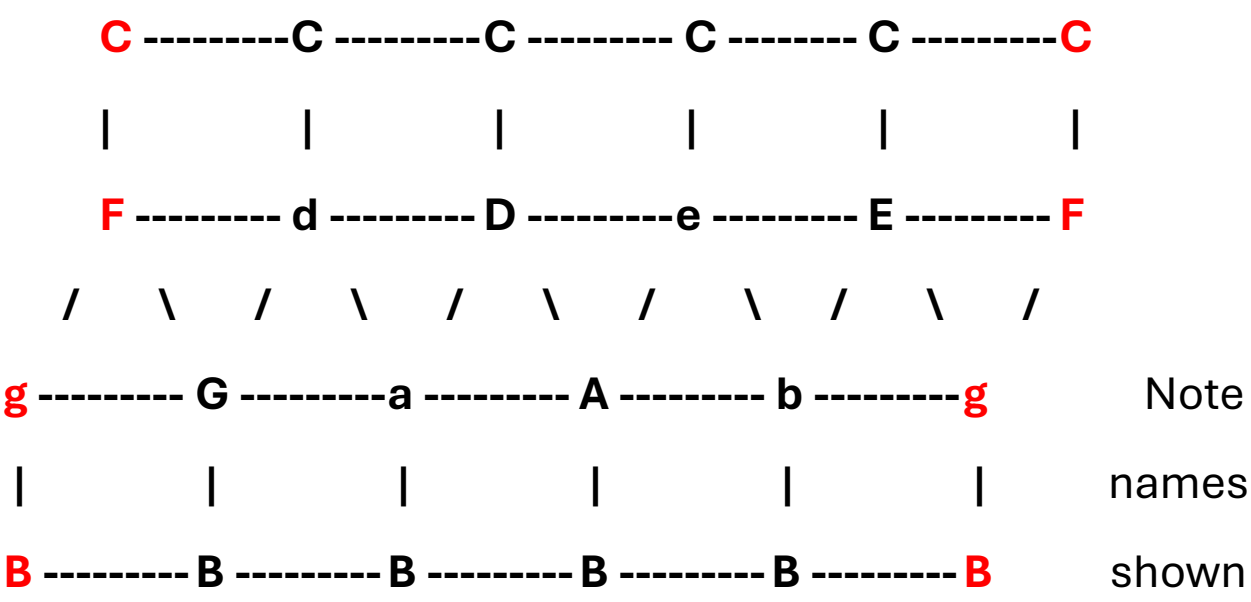
The 20 vertices of the graph are simply indicated via their indices, from 0 to 19; the 10 “side pentagons” of the dodecahedron are the top and bottom pentagons of the dodecahedron; and the 12 keys of the circle-of-fifths are shown in the interior of each pentagon. “Harmonic proximity and similarity” of keys – according to an underlying measure for similarity or distance – must lead to a labelling of the pentagons that (ideally) optimally reflects keys being close to and keys being far from each other. Of the 12! possibilities for labelling the tiles, which labelling with keys is optimal?

The next figure shows a n “unrolled icosahedron.” Its vertices are represented via vertex indices only, from 0 to 11, and the left and right boundary polygons of the sketch represent the periodic boundary, with the same vertex index set. Top and bottom vertices of the icosahedron are shown at the top and bottom in the figure.

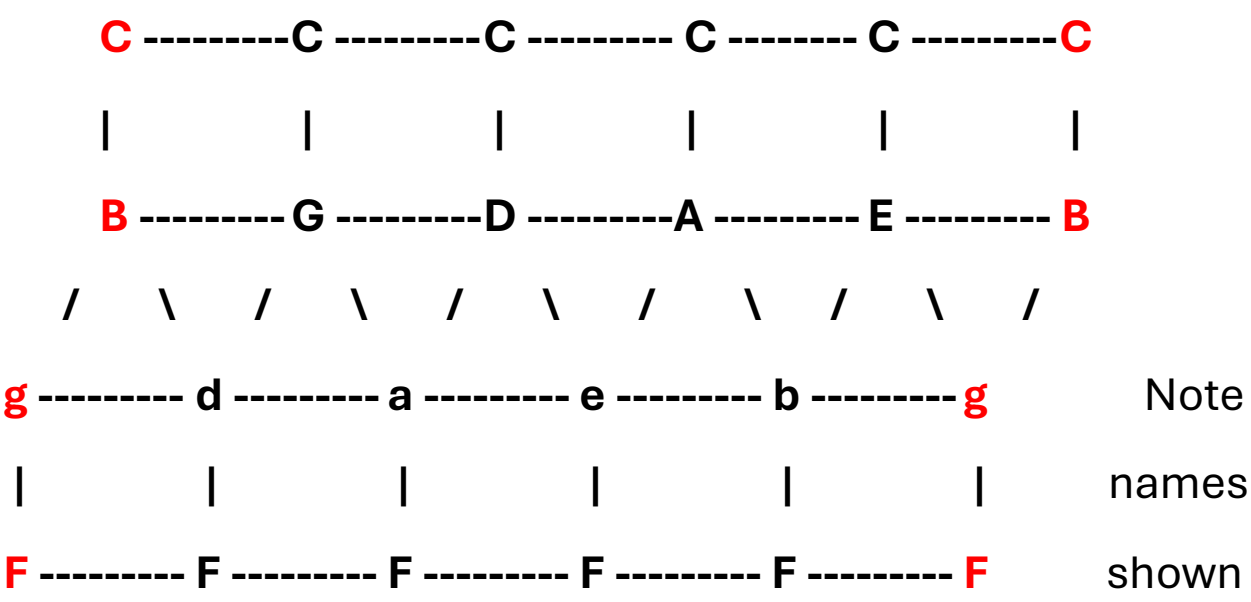


Whenever defining a specific way to associate notes, chords or keys with vertices, edges and tiles, the driving objective is to “**make smooth progressions easily recognizable**.”

For example, the labelling shown in the next figure defines a “path from top to bottom,” including all vertices and notes, where one clearly sees the path CdDeEFgGaAbB using single semi-tone steps when moving forward:

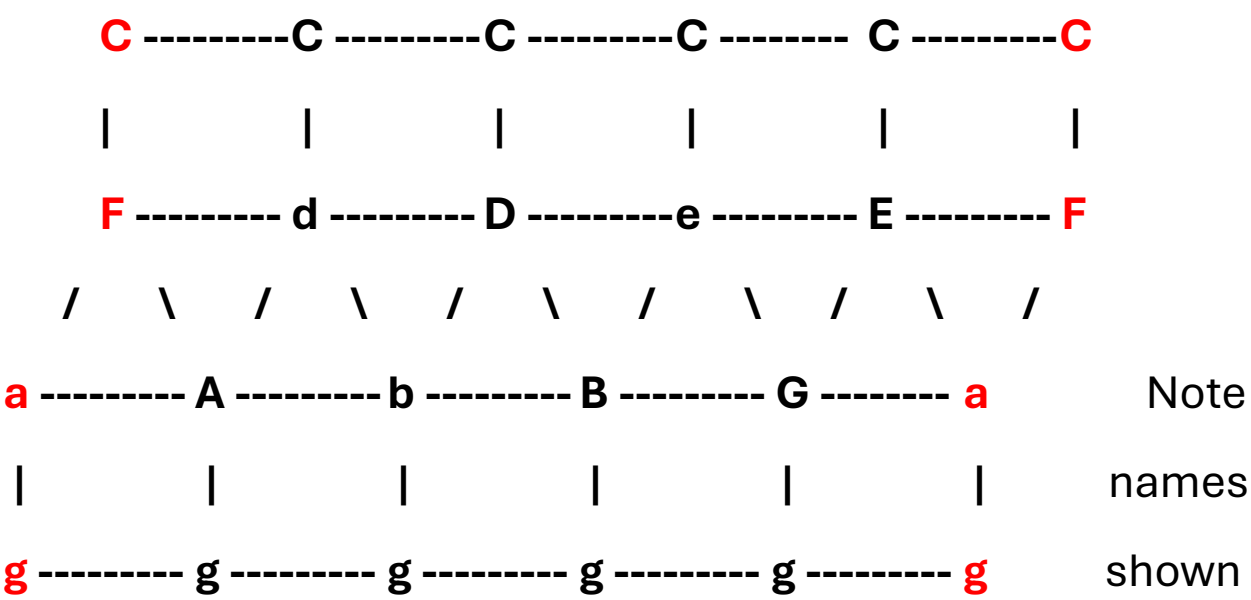


An alternative labelling is shown in the following figure, where the path CGDAEBgdadbF is captured, moving forward seven semi-tones from note to note (from vertex to vertex, top to bottom):



Note. **Projections and Schlegel diagrams** can be used to represent **Hamiltonian cycles** having vertices of Platonic solids.

The icosahedron can be viewed as a structure consisting of six "pairs of vertices that are diagonally opposite to each other" Thus, one can use the note pairs of notes having semi-tone distance six (tritone distance), i.e., the note pairs (C,g), (d,G), (D,a), (e,A}, (E,b) and (F,B), and establish a labelling that reflects these note pairs:

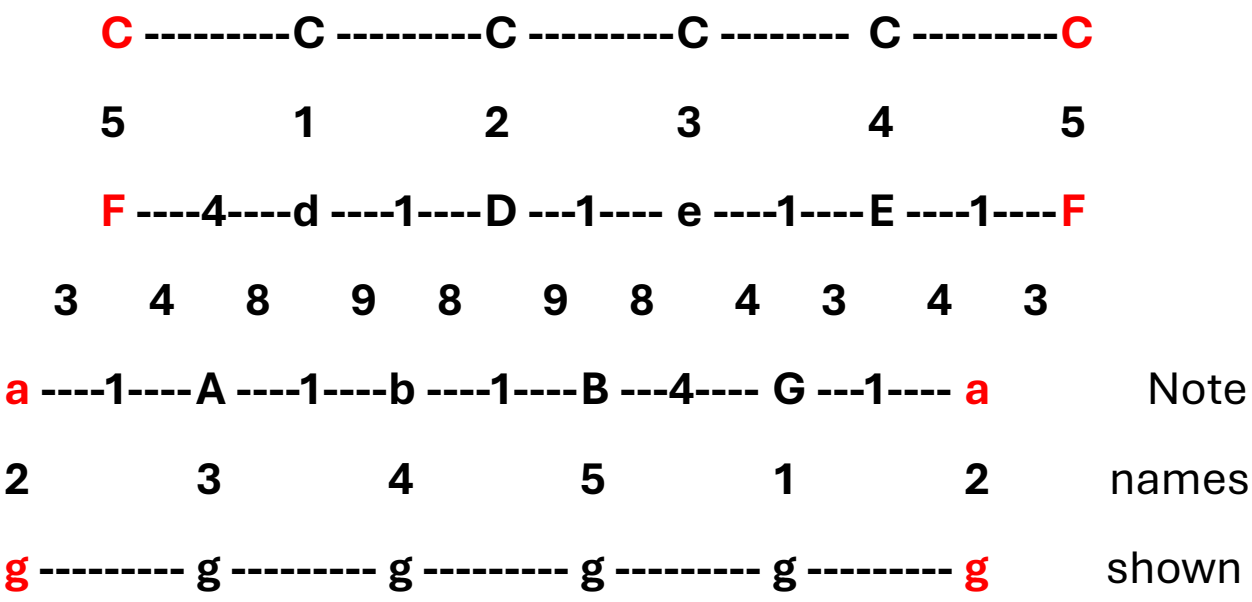


One can associate 4-note paths (and three consecutive edges) with these six pairs, making top-down and left-to-right moves only. For example, these are six of many more possible paths:

- (C,g): C → F → a → g
- (d,G): d → b → B → G
- (D,a): D → B → G → a
- (e,A): e → G → a → A
- (E,b): E → a → A → b
- (F,B): F → A → b → B

Geometrically, one can think of these six pairs as line segments that all pass through the center of the icosahedron and have segment end points that are diagonally opposite to each other.

In terms of note intervals that describe the specific semi-tone intervals associated with edges connecting notes, one can consider the numbers of semi-tone steps one must perform when moving from note to note (using only the semi-tones with indices 0, ..., 11), moving either up (+) or down (-) in the fixed 12-tone chromatic scale C-d-D-e-E-F-g-G-a-A-b-B. The next figure shows these numbers (without the + or – signs) between all note pairs. The numbers are inserted into the middle of the edges. All ten purely downward paths from C to g, for example, lead to a total number of semi-tone steps of 6, properly reflect the tritone interval between C and g. For example, the number 5 is shown on the edge C-F, reflecting that +5 semi-tone steps must be made from C to F and -5 steps from F to C.



Note names shown

We consider a few 4-note paths and compute the total numbers of associated (positively or negatively signed) semi-tone step intervals:

- CFag: +5 +3 -2 = 6 = tritone distance
- CEGg: +4 +3 -1 = 6 = tritone distance
- FAbB: +4 +1 +1 = 6 = tritone distance
- FaGB: +3 -1 +4 = 6 = tritone distance
- DBGa: +9 -4 +1 = 6 = tritone distance
- DbAa: +8 -1 -1 = 6 = tritone distance

The sum is always 6, indicating that the first and last of these 4-note paths indeed represent the tritone note pairs. One achieves this nice result due to the carefully designed labelling of the vertices of the icosahedron with semi-tones.

Note. It is noteworthy that the sum obtained when traversing the three vertices of a tile in a counterclockwise or clockwise order is always zero. Expressed simply, when picking a start vertex, moving from boundary vertex to the next boundary vertex and eventually returning to the start vertex, the sum of all associated note-to-note moves (the interval values mapped to the respective edges) is zero. We consider these three examples: C → F → d → C yields +5 -4 -1 = 0; G → e → B → G yields -4 + 8 -4 = 0; and g → b → A → g yields +4 -1 -3 = 0. (Integral theorems in mathematics and physics are related.)

Intuitively, one might suspect that a semi-tone labelling with the described graph properties holds the potential of capturing various relevant known – and possibly unknown – harmonic relationships. Thus, it sems worthwhile to explore whether interesting chord structures and relationships are represented, e.g., for 3-note and 4-note chords like major and minor triads and standard 7^{th} chords.

As pointed out before, the goal is to use such a graph representation for the recognition, synthesis or study of chord (key) progressions.

We pursue the goal of establishing a harmonically meaningful attachment of semi-tone notes to the icosahedron's vertices. The last couple of visual representations of the icosahedron used essentially "four layers" of vertex groups, i.e., a first, top layer {0}, a second layer {1,2,3,4,5}, a third layer {6,7,8,9,10}, and a fourth, bottom layer {11}. These groups are the four index sets of icosahedron vertex sets that define – also geometrically – the four "vertex layers" of the icosahedron. A labelling objective could be to **use geometrical antipodes**, i.e., diametrically opposite vertices on the sphere in the geometrical sense**, for the placement of** the notes defining the **tritone pairs Cg = (C,g), dG = (d,G), Da = (D,a), eA = (e,A), Eb = (E,b) and FB = (F,B)**. Using the four layers of vertex groups and their associated indices, **the following vertex pairs define antipodes: P0 = (0,11), P1 = (1,9), P2 = (2,10), P3 = (3,6), P4 = (4,7), and P5 = (5,8)**. Abstractly, one can "visualize" these six antipodes in their "geometrical context":

- **0**
- **1 2 3 4 5 (1)**
- **6 7 8 9 10 (6)**
- **11**

Thus, the combinatorial problem one must solve is the problem of determining all possible assignments of tritone pairs to antipodal vertex pairs – subject to certain specified constraints and/or desiderata, to reduce combinatorial complexity.

For example, one could keep fixed the assignment Cg → P0 and consider possibilities for assigning semi-tones to the 2nd and 3rd layers. These are two possible options:

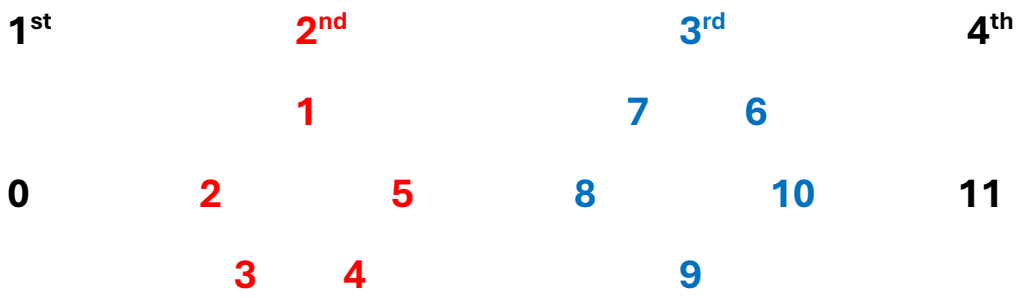
- If one wanted **minimal semi-tone distances between 0 = C and the semi-tones assigned to the 2nd-layer vertices**, one could initially assign the semi-tones in the set {d, D, e, E, F} to this layer, and assign the semi-tones in the set {G, a, A, b, B} to the 3rd–layer vertices.
- If one wanted **maximal semi-tone distances between 0 = C and the semi-tones assigned to the 2nd-layer vertices**, one could initially assign the semi-tones in the set {G, a, A, b, B} to this layer, and assign the semi-tones in the set {d, D, e, E, F} to the 3rd-layer vertices.

Once a decision has been made which of these two alternatives should be used, one must still determine a "proper order" of the chosen five semi-tones for the 2nd layer and the chosen five semi-tones for the 3rd layer. One must also consider how one might have to "rotate" the 2nd and 3rd layers relative to each other, to obtain the desired paths in the representation.

Another "condition" is defined by the desideratum that it should be possible to have two six-vertex, six-edge cycles in the final semi-tone-to-vertex assignment that represent the two unique whole-note scales C-D-E-g-a-b (-C) and d-e-F-G-A-B (-d). There should be two graph cycles with these six vertices and edges having their two vertices labelled with semi-tones being a whole tone apart.

Here, the discussion does not cover in detail the applicable formulae from combinatorics involving factorial computations, reduction of combinatorial complexity being a consequence of symmetry etc. One should simply design and evaluate a chosen labelling and determine whether it provides insights.

A simple “top-down” view of the described four layers of the icosahedron is obtained when looking down from a point above the icosahedron and in the direction 0 (top) → 11 (bottom), where one is positioned somewhere on the line containing the segment 0 → 11. In such a scenario one sees these layers:



Once one has decided which semi-tone set to assign to the 2nd layer and which semi-tone set to assign to the 3rd layer, one can consider 5! = 120 possibilities, for each layer, for ordering and placing the five notes. The additional degree of freedom, i.e., “rotating” the 2nd and 3rd layers relative to each other, provides even more choices. A few of all the possibilities are considered. To make the discussion more compact, assignment possibilities (→) are based on the four-layer model of the icosahedron, which can be visualized like this:

		N0 (C)			→			0		
N1	N2	N3	N4	N5	→	1	2	3	4	5
N6	N7	N8	N9	N10	→	6	7	8	9	10
		N11 (g)			→			11		

One merely needs to select the notes N0, …, N11 from the note set {C, d, D, …, B} and create an ordered 12-tuple that defines a note-to-vertex assignment. These are two examples that keep in mind the tritone pairs and their associated antipodes, as discussed above:

- (C, d, D, e, E, F, A, b, B, G, a, g)
- (C, G, a, A, b, B, e, E, F, d, D, g)

A decision to be made concerns the/an "ideal" order of the five notes to be placed on the 2nd- and 3rd -layer vertices. One must ask whether it is advantageous to place notes with small or with large semi-tone distance next to each other on the five-note cycles; one must ask which note sub-set to select for placement on the 2nd and 3rd vertex layers; and one must ask whether to rotate the 2nd and 3rd layers relative to each other. A guiding principle for note-to-vertex assignments could be the following:

> **One can assign semi-tones to icosahedron vertices such that the interval distances between pairs of semi-tones are captured and reflected by corresponding distance values between vertex pairs to which the semi-tones are assigned.**
>
> **Distance of two vertices** of the icosahedron would be defined by the minimal number of edges needed to connect the two vertices. A **cost function to be optimized for an optimal note-to-vertex assignment** could be a function that minimizes the difference between the prescribed or expected distances of vertex pairs for specified note pairs and the actual, real distances seen for the vertex pairs that have those specified note pairs assigned to them in the devised or computed graph.

As the size of the search space to be explored for this combinatorial problem is still "rather small," it should be permissible to determine and evaluate all possible note-to-vertex assignments and identify those that have optimal cost function value.

In the following, attempts are described for assigning notes to vertices that mainly rely on intuition and simple guiding principles:

- The **whole-tone cycle C-D-E-g-a-b (-C)** can be interpreted as an "ideal" cycle to be "preserved" on the icosahedron, by assigning subsequent notes in the whole-tone-cycle to subsequent vertices on six-vertex cycle.
- The **whole-tone cycle d-e-F-G-A-B (-d)** can be interpreted as a complementary "ideal" cycle to be "preserved" as well on the icosahedron, by assigning subsequent notes in the whole-tone-cycle to subsequent vertices on a six-vertex cycle.
- The integer **index sequences** used for the notes in these two cycles are **0-2-4-6-8-10 and 1-3-5-7-9-11**, exhibiting the "highly desirable characteristic" that all pairs of two consecutive index values have the absolute value two.
- The two-note pairs defining the six **tritone pairs (C,g), (d,G), (D,a), (e,A), (E,b) and (F,B)** should be assigned to vertices of the icosahedron such that the **shortest paths connecting the two notes of each tritone pair always have three edges**. Further, every vertex pair having the notes of a **tritone pair** assigned to it should be an antipodal pair of vertices on the (spherical geometry of the) icosahedron.

These facts, principles and desired properties should be the basis of a strategy for assigning notes to icosahedron vertices. We can try to construct possible assignments. Here, we concatenate names of notes and assigned vertex indices, to be more compact:

1st assignment: **C0, D1, E2, g3, a4, b5, d6, e7, F8, G9, A10, B11**
2nd assignment: **C0, G1, D2, b3, a4, F5, B6, A7, E8, a9, d10, g11**

3rd assignment: **C0, d1, D2, e3, E4, F5, A6, b7, B8, G9, a10, g11**

These assignments can be shown in their geometrical context, i.e., shown in an abstract four-layer representation of the icosahedron:

1st

			C0		
	D1	E2	g3	a4	b5
d6	e7	F8	G9	A10	
		B11			

The two **whole-tone cycles are represented** by the six top and the six bottom vertices, respectively. Edges in the cycles connect two subsequent cycle notes. *Vertices with two notes defining a tritone pair do not have the consistent edge distance of three.*

2nd

			C0		
	G1	D2	b3	e4	F5
B6	A7	E8	a9	d10	
		g11			

The two **whole-tone cycles are represented** by the two groups of six vertices each, shown in two colors. Edges in the cycles connect two subsequent cycle notes. *Vertices with two notes defining a tritone pair do not have the consistent edge distance of three.*

3rd

			C0		
	d1	D2	e3	E4	F5
A6	b7	B8	G9	a10	
		g11			

All **tritone pairs have minimal edge distance of three**. The notes of all **tritone pairs have antipodal vertices** of the icosahedron.

Whole-note cycles are not properly defined by this assignment, i.e., consecutive cycle notes are not edge-connected by the underlying connectivity.

Does a note-to-vertex assignment exist that satisfies all desiderata? We consider a 4th assignment, based on the circle of fifths defining the assignment orders of notes to vertices in layer 2 (G, D, A, E, B) and layer 3 (F, b, e, a, d):

4th assignment: **C0, G1, D2, A3, E4, B5, e6, b7, F8, d9, a10, g11**

4th			**C0**			All **tritone pairs have minimal edge**
	G1	**D2**	**A3**	**E4**	**B5**	**distance of three**. The notes of all
e6	**b7**	**F8**	**d9**	**a10**		**tritone pairs have antipodal**
		g11				**vertices** of the icosahedron.

Whole-note cycles are not properly defined by this assignment, i.e., consecutive cycle notes are not edge-connected by the underlying connectivity.

Considering the described four possible assignments, none of them satisfies all the stated desiderata for an ideal, optimal assignment that would have all six tritone pairs placed on antipodal vertices of the icosahedron and simultaneously have both whole-tone cycles, CDEgab(C) and deFGAB(d), represented by two (closed) polygons of length six with two consecutive notes in the whole-tone cycles being end points/vertices of edges that exist as edges of the icosahedron.

Graph theory provides graph analyses and characterizations -- with formal proofs -- one can use for devising note-to-vertex assignments being harmonically useful and effective for conveying progressions.

In fact, Imai et al. [5] determined four assignments that satisfy all the desiderata listed above and, in addition and consequently, are assignments that map the ordered sequence of the 12 semi-tones of the chromatic scale to the ordered sequences of the vertices of closed 12-vertex polygons on the icosahedron, i.e., consecutive semi-tones are mapped to consecutive vertices of the polygons. These are the four assignments presented by Imai et al. [5]:

5th assignment: **C0, D1, e2, B3, b4, d5, F6, E7, G8, a9, A10, g11**
6th assignment: **C0, D1, d2, A3, b4, B5, e6, E7, F8, a9, G10, g11**
7th assignment: **C0, d1, e2, E3, a4, B5, b6, D7, F8, G9, G10, g11**
8th assignment: **C0, d1, E2, a3, A4, B5, D6, e7, F8, G9, b10, g11**

5th

C0

D1 e2 B3 b4 d5

F6 E7 G8 a9 A10

g11

6th

C0

D1 d2 A3 b4 B5

e6 E7 F8 a9 G10

g11

7h

C0

d1 e2 E3 a4 B5

b6 D7 F8 G9 A10

g11

8th

C0

d1 E2 a3 A4 B5

D6 e7 F8 G9 b10

g11

One should ask important questions concerning this “embedding of scales and chords” and its **readability**: (1) Does one “see” the seven notes of the **diatonic scale**? (2) Does one “see” **chords**, especially major/minor triads and 7th chords? Imai et al. [5] discuss golden-ratio triangles, defined by icosahedron vertices, to identify triads.

It is rather amazing that only these four assignments satisfy the "design objective" for representing tritones as antipodes and having the 12-tone chromatic scale captured in all four assignments and the six-note whole-tone scale CDEgab in assignments 5 and 6 and the six-note whole-tone scale deFGAB in assignments 7 and 8. Of course, one can consider many other assignments and explore their use for capturing known or otherwise interesting structures.

Imai et al. [5] also consider assignments that order the 12 semi-tones according to the circle-of-fifths (Pythagorean) order. Thus, the order CdDeEFgGaAbB(C) is replaced by CGDAEBgdaebF(C). The goal becomes to capture the circle-of-fifths order as 12-vertex polygons on the icosahedron. The four new assignments are:

9th assignment: **C0, D1, A2, F3, b4, G5, B6, E7, d8, a9, e10, g11**
10th assignment: **C0, D1, G2, e3, b4, F5, A6, E7, B8, a9, d10, g11**
11th assignment: **C0, G1, A2, E3, a4, F5, b6, D7, B8, d9, e10, g11**
12th assignment: **C0, e1, F2, G3, E4, a5, d6, b7, D8, A9, B10, g11**

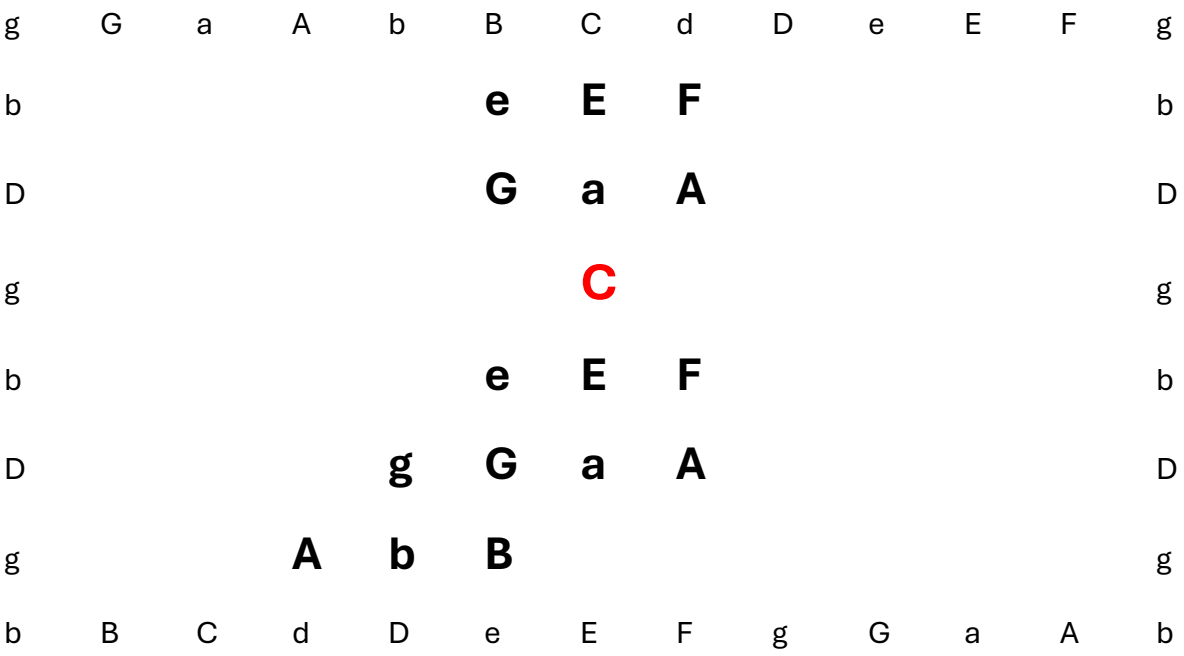
9th			**C0**		
	D1	**A2**	**F3**	**b4**	**G5**
B6	**E7**	**d8**	**a9**	**e10**	
		g11			

10th			**C0**		
	D1	**G2**	**e3**	**b4**	**F5**
A6	**E7**	**B8**	**a9**	**d10**	
		g11			

11th			**C0**		
	G1	**A2**	**E3**	**a4**	**F5**
b6	**D7**	**B8**	**d9**	**e10**	
		g11			

12th			**C0**		
	e1	**F2**	**G3**	**E4**	**a5**
d6	**b7**	**D8**	**A9**	**B10**	
		g11			

Once again, we use a part of the representation of the Tonnetz, without explicitly showing connections (edges) between notes, to emphasize the concept of placing harmonically close notes closely in a specific arrangement in the plane. The following figure merely shows the note names, placed in an arrangement that emphasizes implied chords. One must “read” the notes in a top-down fashion, and possibly in a left-to-right way, to identify relevant chords.



One must “read” this figure (with small-font notes simply defining a frame) in a top-down fashion and possibly in a left-to-right manner.

First, in the **first line**, one sees the notes e, E and F. **Moving downward** from them by **two lines** one sees these triads:

- Cmin = (e,G,C), amaj = (e,a,C), Cmaj = (E,G,C), aaug = (E,a,C), Amin = (E,A,C), Fmin = (F,a,C) and Fmaj = (F,A,C).

Second, in the **second line**, one sees the notes G, a and A. **Moving downward** from them by **two lines** one sees the following triads and one (incomplete) 9^{th} chord:

- Cmin = (G,C,e), Cmaj = (G,C.E), F9 = (G,C,F) (circle-of-fifths notes), amaj = (a,C,e), aaug = (a,C,E), Fmin = (a,C,F), Adim = (A,C,e), Amin = (A,C,E) and Fmaj = (A,C,F).

Third, in the **third line**, one sees the note C. **Moving downward** from it by **two lines** defines triads, while **moving downward** by **three lines** defines 7^{th} chords. **Moving downward** by **two lines and then from left to right** defines 6^{th} chords. One sees these chords:

- Cdim = (C,e,g), Cmin = (C,e,G), amaj = (C,e,a), Cmaj = (C,E,G), Caug = (C,E,a), Amin = (C,E,A), Fmin= (C,F,a), Fmaj = (C,F,A),
- Cdim7 = (C,e,g,A), Chdim7 = (C,e,g,b), Cmin7 = (C,e,G,b), C7 = (C,E,G,b), Cmaj7 = (C,E,G,B) and
- Cminb6 = (C,e,G,a), Cmin6 = (C,e,G,A), Cmajb6 = (C,E,G,a) and Cmaj6 = (C,E,G,A).

Here, the note C is at the center of a local "group" that defines these sets of triads (in root positions and inversions), 7^{th} chords, 6^{th} and 9^{th} chords. If one were to expand this local "group" by adding more notes to the left and to the right of it, one could also recognize 7^{th} chord inversions s in the expanded representation, for example. It is now possible to apply the same "note arrangement template" to any note serving in place of the note C. One can then recognize the same chords, relative to new roots, in the same template locations.

Note. For simplification of the template, one can ignore the first line (e, E, F) and second line (G, a, A) for downward chord identification,

as relevant chords are captured by moving downward from the third line (C). One could write the simplified template as a set of note sets, i.e., {C, {e,E,F}, {g,G,a,A}, {A,b,B} }. In order to make the template independent of specific notes, one can write it in general note-index notation as the set of integer sets { 0, {3,4,5}, {6,7,8,9}, {9,10,11} }.

By "shifting" this template to the left and to the right, for example, one obtains the note arrangements showing the relevant chords associated with all possible root notes. In the figure, the template is applied simultaneously to root notes F, C and G.

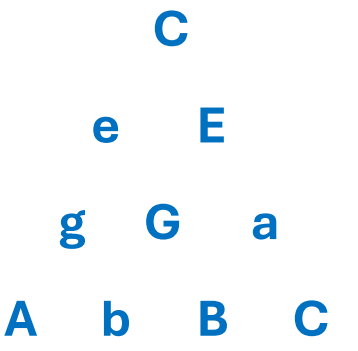
< ---------- Circle-of-Fifths Root Notes ---------->

			F				C				G	
		a	A	b		e	E	F		b	B	C
	B	C	d	D	g	G	a	A	d	D	e	E
D	e	E		A	b	B		E	F	g		

By traversing the notes residing inside the template regions under the root notes F, C and G, one obtains all relevant chords of interest for these root notes, being neighbors in the circle of fifths. Common suspended chords (sus2, sus4) are not included in this discussion.

Note. Paths that only move diagonally left, downward determine diminished chords (e.g., diminished triads and fully diminished 7^{th} chords). Vertical, downward paths determine augmented triads.

We now return to the note sub-tree that represents 7^{th} chords.



The left triangle represents several 4-note chords, including: C dim7 = (C,e,g,A), C hdim7 = (C,e,g,b), C min7 = (C,e,G,b), C 7 = (C,E,G,B), and C maj7 = (C,E,G,B).

Thus, this local set of notes in the tree contains the eight notes that define the five 7th chords Cdim7, Chdim7, Cmin7, C7 and Cmaj7.

For key/scale independence, one uses integers. Generally, one can identify the chromatic scale notes with integer values as follows:

0

3 4

6 7 8

9 10 11 0

0 1 2 3 4

3 4 5 6 7 8

6 7 8 9 10 11 0

9 10 11 0 1 2 3 4

0 1 2 3 4 5 6 7 8

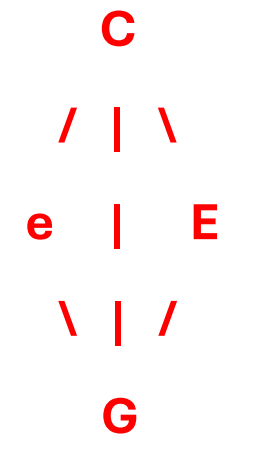
3 4 5 6 7 8 9 10 11 0

6 7 8 9 10 11 0 1 2 3 4

9 10 11 0 1 2 3 4 5 6 7 8

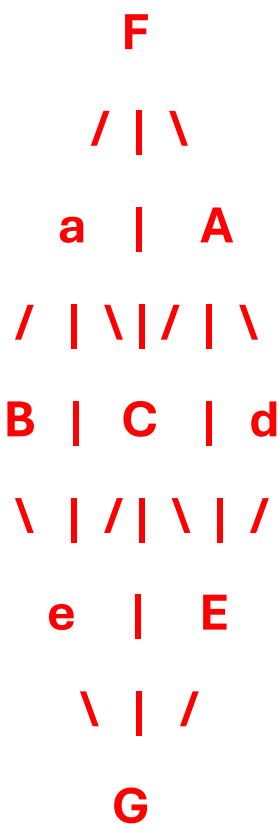
0 1 2 3 4 5 6 7 8 9 10 11 0

Note. Sometimes the letters **‘t’ and ‘e’** are used as substitutes for the integer values **ten and eleven**, respectively.



This visualization of the C maj and C min triads captures the six possible note permutations (positions), e.g., possible C maj triad positions (C,E,G), (E,G,C), and (G,C,E) and (C,G,E), (G,E,C) and (E,C,G).

The first three of these triads are the root position and the two inversions of the C maj triad, obtained by traversing the triangle vertices in clockwise order in thia figure. Traversing the graph’s vertices in counterclockwise order yields the other three positions.



The left excerpt shows the six triangles sharing the note C. The triangles define triads C maj = (C,E,G), A min = (A,C,E), F maj = (F,A,C), F min = (F,a,C), a maj = (a,C,e) and C min = (C,e,G).

Smooth triad progressions are obtained by moving from a “start” to a “goal” triangle, i.e., by moving from triangle to a neighbor triangle via shared edges (defining two shared triad notes).

Moving from triangle to triangle (triad to triad) via the shared edge (two shared notes) leads to smoother progressions than moving from triangle to triangle by preserving only one note (or note at all). For example, one can define this smooth triad progression from amin = (a,B,e) to Amaj=(A,d,E) by the following path: amin=(a,B,e) → amaj=(a,C,e) → Cmin=(C,e,G) → Cmaj=(C,E,G) → Amin=(A,C,E) → Amaj=(A,d,E). Of course, one can play these triads in root position of one of their inversions.

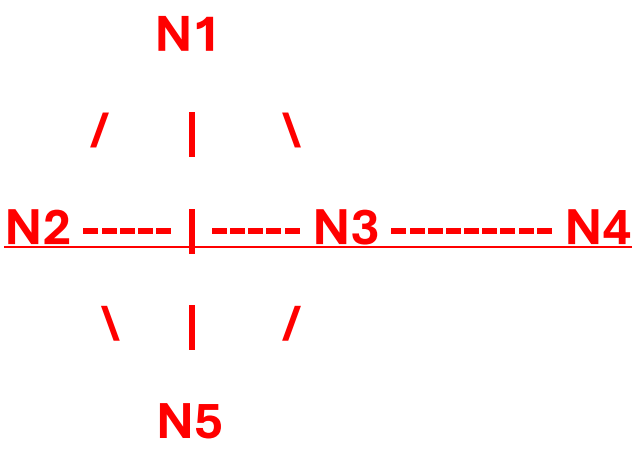
Considering the binary tree representation that only uses integer values, the relevant number sequences in the tree (defining lines and line segments establishing vertex connectivity in local neighborhoods of a graph structure implied by the tree) are:

- **Left edge (downward)**: 0, 3, 6, 9, 0, ..., e.g., C, e, g, A, C, ...
  = circle of minor thirds: (+3) mod 12.
- **Right edge (downward)**: 0, 4, 8, 0, ..., e.g., C, E, a, C, ...
  = circle of major thirds: (+4) mod 12.
- **Vertical line (downward)**, center line:
  0, 7, 2, 9, 4, 11, 6, ..., e.g., C, G, D, A, E, B, g, ...
  = circle of fifths: (+7) mod 12.
- **Horizontal line (left to right),** bottom of tree:
  0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 0, e.g., C, d, ..., B, C
  = circle of minor seconds: (+1) mod 12.
- **Horizontal line (left to right, every second note/number)**, bottom of tree: 0, 2, 4, 6, 8, 10, 0, e.g., C, D, E, g, a, b, C

= circle of major seconds: (+2) mod 12.

- **Left edge (upward)**: 0, 9, 6, 3, ..., e.g. C, A, g, e, ...
  = circle of major sixths: (+9) mod 12.
- **Right edge (upward)**: 0, 8, 4, 0, ..., e.g., C, a, E, C, ...
  = circle of minor sixths: (+8) mod 12.
- **Vertical line (upward)**, center line:
  ... 6,11, 4, 9, 2, 7, 0, e.g., ..., g, B, E, A, D, G, C
  = circle of fourths: (+5) mod 12.
- **Horizontal line (right to left)**, bottom of tree:
  0, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, e.g., C, B, ..., d, C
  = circle of major seconds: (+11) mod 12.
- **Horizontal line (right to left, every second note/number)**, bottom of tree: 0, 10, 8, 6, 4, 2, 0, e.g., C, b, a, g, E, D, C
  = circle of minor sevenths: (+10) mod 12.

One can express these modulo-12 (called mod 12) operations more compactly as follows (top-to-bottom and left-to-right directions):



N2 = (N1+3) mod 12

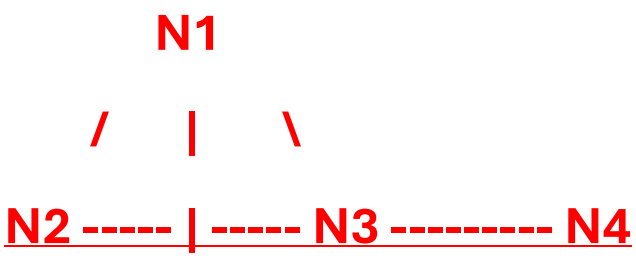
N3 = (N1+4) mod 12

N5 = (N1+7) mod 12

N3 = (N2+1) mod 12

N4 = (N2+2) mod 12

When considering modulo-12 operations performed using the bottom-to-top and right-to-left directions one gets:



N1 = (N2+9) mod 12

N1 = (N3+8) mod 12

N1 = (N5+5) mod 12

N2 = (N3+11) mod 12

N2 = (N4+10) mod 12

Note. Suspended 3-note chords/triads are captured in the binary tree (a note graph) as well. Consider Csus2 = (C,D,G) and Csus4 = (C,F,G), for example. One finds the Csus2 chord by starting at C, moving to the right by two semi-tones (+2) and moving vertically up by five semi-tones (+5), modulo 12, in general. One finds the Csus4 chord by starting at C, moving up vertically by five semi-tones (+5) and moving to the right by two semi-tones (+2), modulo 12, in general.

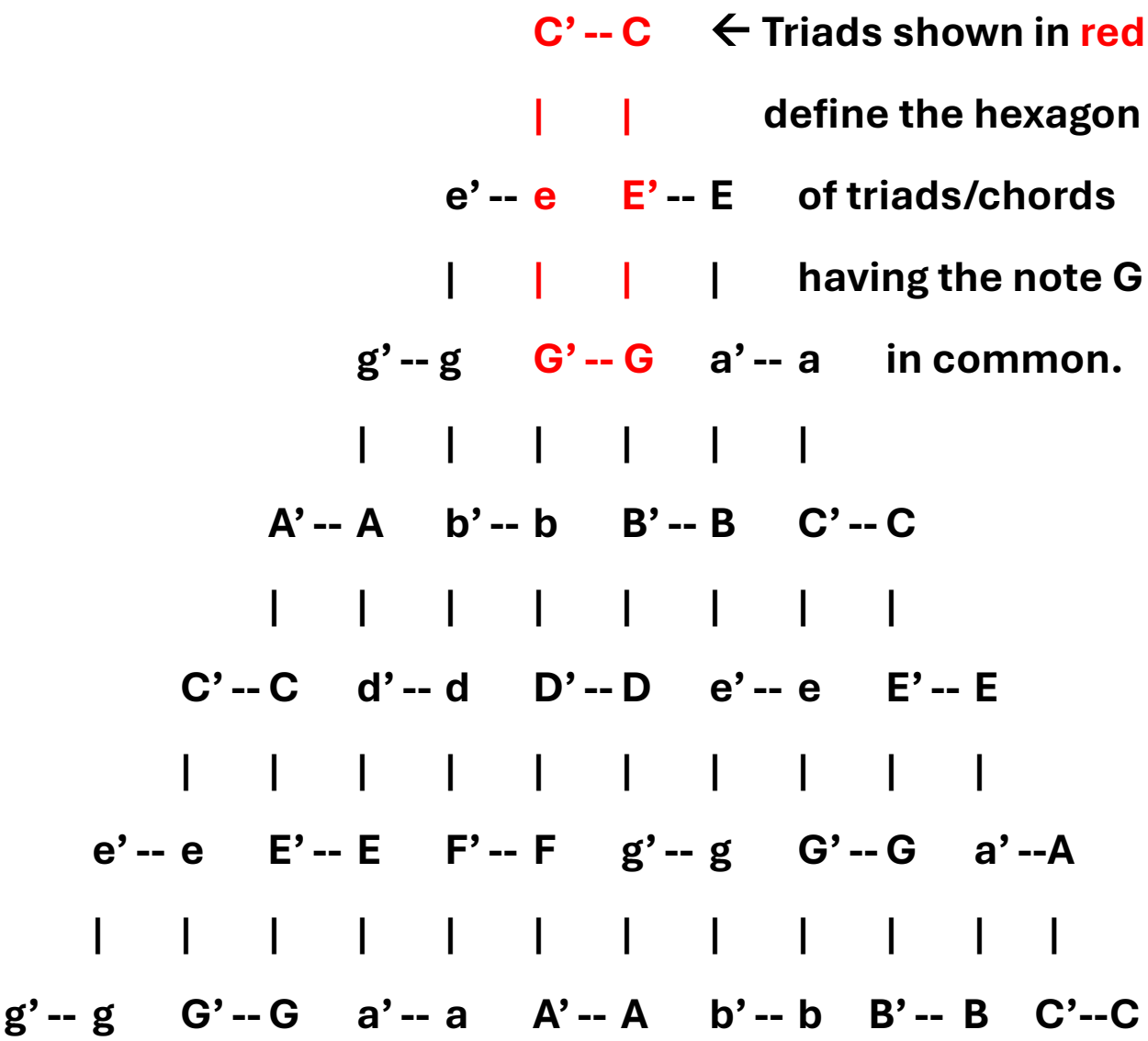
Further, one can turn the note-based binary tree (graph) into a dual chord-/triad-based graph by turning triangles given by the three notes defining a chord/triad into a vertex of a dual graph. These new "chord vertices" can now be connected when they lie in neighboring triangles, i.e., when two notes are shared.

To be compact and save space, we use an non-standard notation: A small letter indicates a flattened note, and the symbol ' refers to a minor chord, i.e.:

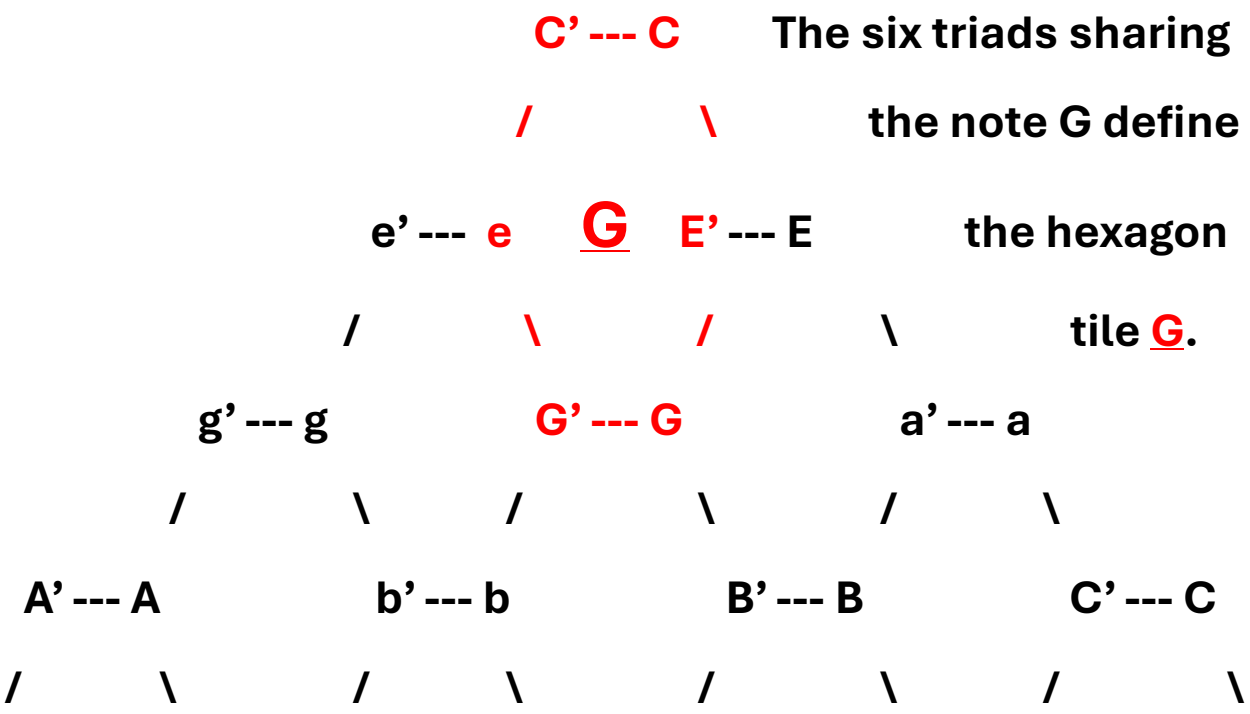
Ab min = a' = (a, c, e), **Ab maj = a = (a,C,e),**

A min = A' = (A,C,E), **A maj = A = (A,C#.E), ...**

The resulting chord-based graph is sketched here:



The following representation -- geometrically slightly different -- shows more clearly that all "chord vertices" in this dual graph (dual to the binary note tree tree) have vertex rank (valence) three, with each "chord vertex" surrounded by three hexagonal (Voronoi) tiles



The chord graph depicted here is correct: At the top of the graph, one sees the hexagonal region ("chicken-wire region") of the six-chord sequence C maj → E min → G maj → G min → Eb maj → C min. Spelling these triads out, they are C = (C,E,G), E' = (E,G,B), G = (G,B,D), G' = (G,Bb,D), e' = (Eb,G,Bb) and C' = (C,Eb,G). These six chords – all being 3-note major or minor triads -- define the six chords sharing the note G. The red hexagonal tile in the figure highlights the **G** tile. This visualization of the dual chord graph shows a plane tessellation with hexagonal tiles – a so-called **Voronoi complex** that is **dual to** the note-based so-called **Delaunay complex** consisting of triangles (in this setting).

This dual triad/chord graph has triads as vertices and connects vertices when the respective triads have two

common notes – implying that a smooth progression is possible (in both directions “across” the connecting edge).

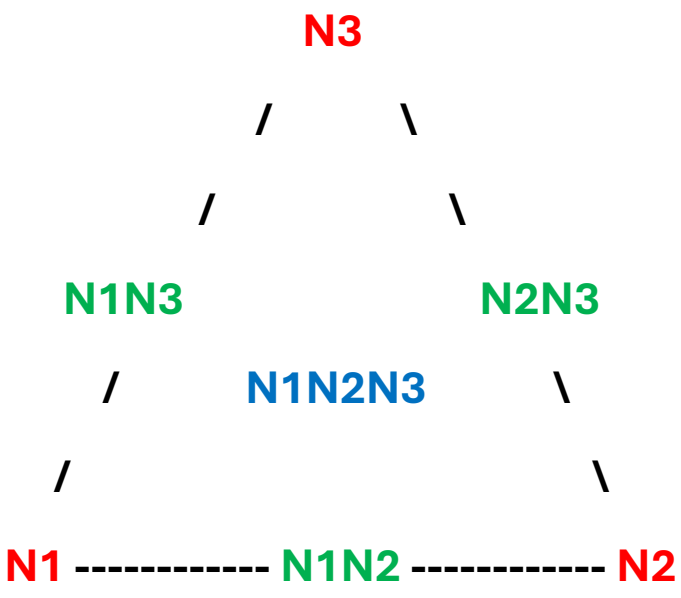
One can also interpret such dual graphs – note graphs and chord graphs – as “partial projections” e.g., (into the plane) of more general complexes (e.g., simplicial complexes) consisting of 0-dimensional (0D) vertices, 1D line segments/edges, 2D triangles, 3D tetrahedra etc. For example, a 3-note chord maps to three vertices: notes N1, N2 and N3 – defining the three vertices v1, v2 and v3 of a triangle, respectively. Since the three notes are assumed to be played “simultaneously” they produce “seven sounds,” i.e., the sounds of N1, N2 and N3 individually (vertices), the 2-note chord sounds of N1N2, N1N3 and N2N3 (edges), and the 3-note chord sound N1N2N3 (triangle). At this point, we ignore permutations of these notes and chord positions and inversions here. Thus, the implied 2D simplex (triangle) represents 2^3-1=7 sounds. (If one also counted the “no-sound chord,” the number of sounds would be 2^3=8.)

Analogously, a 3-note chord maps to four vertices: notes N1, N2, N3 and N3 – defining the four vertices v1, v2, v3 and v4 of a tetrahedron, respectively. Since the four notes are assumed to be played “simultaneously” they produce “15 sounds”: the sounds of N1, N2, N3 and N4 individually (vertices), the 2-note chord sounds of N1N2, N1N3, N1N4,

N2N3, N2N4 and N3N4 (edges), the 3-note chord sounds of N1N2N3, N1N2N4, N1N3N4 and N2N3N4 (faces/triangles), and the 4-note sound of N1N2N3N4 (tetrahedron). Again, we ignore permutations of these notes and positions and chord inversions. Thus, the implied simplex represents 2^4-1=15 sounds. (If one counted the “no-sound chord,”, the number of sounds would be 2^4=16.)

ONE CAN NOW CONSIDER DUALITY AS ONE DOES, FOR EXAMPLE, WHEN DUALIZING A DELAUNAY COMPLEX AND A VORONOI COMPLEX. WE CONSIDR THE CASE OF A TRIANGLE WITH VERTICES V1, V2 AND V3. ONE CAN DUALIZE EACH OF ITS CONSTITUENT ELEMENTS, I.E., ONE CAN DUALIZE ITS VERTICES, ITS EDGES AND THE TRIANGLE ITSELF. THE DUAL OF A VERTEX IS THE SAME VERTEX (V ←→ V); THE DUAL OF AN EDGE IS THE EDGE’S “MID-POINT” (V1V2 ←→ MID-POINT (V1,V2) ETC.) AND THE DUAL OF THE TRIANGLE IS ITS “CENTER-POINT” (V1V2V3 ←→ CENTER-POINT (V1,V2,V3)). ONE CAN GENERALIZE THIS CONCEPT TO A TETRAHEDRON AND EVEN HIGHER-DIMENSIONAL SIMPLICES TO REPRESENT GENERAL K-NOTE CHORDS, I.E., TO K-NOTE NOTE COMPLEXES AND THEIR DUALS OBTAINED BY DUALIZING THE CONSTITUENT ELEMENTS OF A K-NOTE COMPLEX. IF ONE DUALIZES A 4-NOTE-BASED TETRAHEDRON, ONE GENERATES AS DUAL ELEMENTS MID-POINTS OF ITS EDGES, CENTER-POINTS OF ITS TRIANGLE FACES AND ITS AVAERAGE POINT OF ALL FOUR VERTICES.

One can represent the 3-note case in the plane like this:



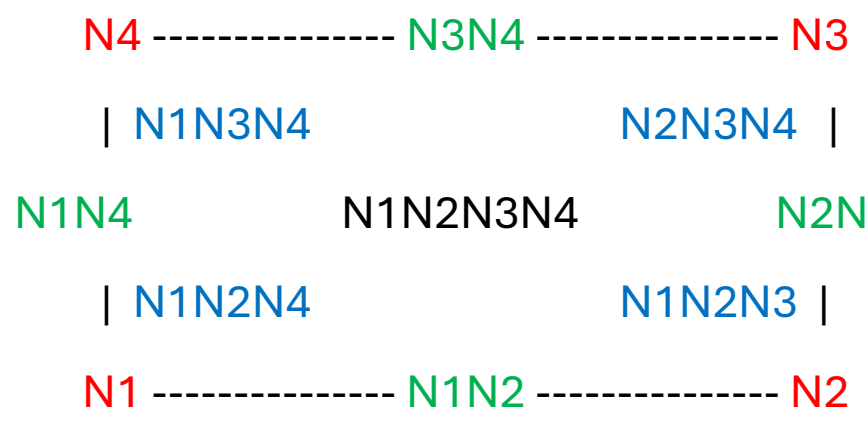
red: 1-note vertices

green: 2-note edges

blue: 3-note triangle

There are 0-, 1- and 2-dimensional sub-simplices in this chord illustration.

Projecting and visualizing a 4-note chord in the plane more difficult. The following illustration attempts to provide a meaningful "abstract visualization" of the 3-simplex, i.e., a tetrahedral configuration:



Intersecting diagonal edges are missing in this projection into the plane.

This projection of the 4-note chord tetrahedron includes all 0-, 1-, 2- and 3-dimensional sub-simplices representing all possible 1-, 2-, 3- and 4-note chords, respectively. Subject to the indexing of the vertices/notes, one obtains different orientations of the sub-simplices and therefore the different permutations, positions and inversions of all the chords, including the "standard" root positions

and inversions of all chords. Focusing on the notes' indices, the possibilities are:

- 1, 2, 3, 4 (four "1-note chord" permutations)
- 12, 21, 13, 31, 14, 41, 23, 32, 24, 42, 34, 43
  (12 2-note chord permutations)
- 123, 132, 231, 213, 312, 321, 124, 142, 241, 214, 412, 421, 134, 143, 341, 314, 413, 431, 234, 243, 342, 324, 423, 432 (24 3-note permutations)
- 1234, 1243, 1324, 1342, 1423, 1432, 2134, 2143, 2314, 2341, 2413, 2431, 3124, 3142, 3214, 3241, 3412, 3421, 4123, 4132, 4213, 4231, 4312, 4321
  (24 4-note permutations)

The total number of these chord permutations, 64, is a sum of multiplied binomial coefficients, written as 4Ck, k = 1...4:

SUM [k! * (4 choose k)] = SUM [k! * (4Ck)] =
= 1 * (4C1) + 2 * (4C2) + 6 * (4C3) + 24 * (4C4) =
= 1 * 4 + 2 * 6 + 6 * 4 + 24 * 1 = 4 + 12 + 24 + 24 = **64**.

Not all 64 inversions (index permutations) are generally of interest, harmonically, and the following chord possibilities might suffice:

- 1, 2, 3, 4 (four "1-note chord" selections);
- 12, 13, 14, 23, 24, 34 (six 2-note chord selections);
- 123, 124, 134, 234 (four 3-note permutations); and
- 1234 (one 4-note selection).

The number of these chords is only 2^4-1 = **15**. Regardless, from a musical perspective, positions and inversions of chords are important, as they produce distinctly different sounds with varying

effects -- of the same harmony. Thus, positions and inversions of chords represent “harmonically equivalent yet musically separate chord sound instances.”

Two important questions arise regarding geometrical aspects:

> **(1)** HOW DOES ONE DEFINE THE LENGTHS OF EDGES IMPLIED BY TWO NOTES N1 AND N2, I.E., HOW DOES ONE ASSOCIATE SPATIAL COORDINATES WITH THE RESPECTIVE VERTICES V1 AND V2, AND WHAT IS “DISTANCE”?
> **(2)** HOW DOES ONE DEFINE, PROJECT, VISUALIZE AND INTERPRET SUCH PROJECTED SIMPLICIAL COMPLEXES, ORIGINALLY EMBEDDED IN A HIGH-DIMENSIONAL SPACE, IN THE “POJECTION SPACE,” E.G., IN A PLANE”?

We consider the specific (fully) diminished 7^{th} chord C dim7 = C-Eb-Gb-Bbb = C-Eb-Gb-A, keeping in mind that the integer values associated with its four notes are 0, 3, 6 and 9. One can map the four notes to the vertices of a tetrahedron: C→ v0, Eb → v1, Gb → v2, A → v3. When one wants to embed this tetrahedron geometrically in 3D space one must devise a method for determining coordinate triples (x,y,z) for each of the four vertices. One could utilize the fact that all pairs of consecutive notes of this (cyclic) chord have the same integer semi-tone distance of 3. In other words, the distances in geometric space could be defined such that they satisfy the edge length condition ||v1-v0|| = ||v2-v1|| = ||v3-v2|| = 3. Further, considering the associated note integer values again, one can enforce the constraint ||v2-v0|| = ||v3-v1|| = 6 and ||v3-v0|| = 9.

What is the resulting geometry of the implied tetrahedron in 3D space?

What makes the (fully) diminished 7th chord, e.g., C dim7 = CEbGbBbb = CEbGbA, special? This chord is special since its 3-, 2- and 1-note subsets are 3-, 2- and 1-note subsets of many 3-note chords (and many other chords). Using this 7th chord's scale-independent note integer tuple (0,3,6,9), one obtains these neighborhoods – considering at this point only 3-note chords/triads as neighbors:

- The chord's 1-note triad neighbors share 0, 3, 6 or 9.
- Its 2-note triad neighbors share (0,3), (0,6), (0,9), (3,6), (3,9) or (6,9).
- Its 3-note triad neighbors share (0,3,6), (0,3,9), (0,6,9) or (3,6,9).

Specifically, when using 0=C one obtains these neighbors:

- This7th chord's 3-note neighbor chords sharing exactly three notes are (0,3,6) = CEbGb = C dim, (3,6,9) = EbGbBbb = Eb dim, (6,9,0) = GbBbbC = Gb dim and (9,0,3) = ACEb = A dim, all being diminished triads.
- This 7th chord's major and minor triad neighbors sharing exactly two notes are (0,3,7) = CEbG = C min, (0,3,8) = CEbAb =Ab maj, (3,6,10) = EbGbBb = Eb min, (3,6,11) = EbGbB = D#G#B = B maj, (6,9,1) = GbBbbDb =F#AC# = Gb min = F# min, and (6,9,2) = GbBbbD = F#AD = D maj.
- This 7th chord's 3-note chord neighbors sharing exactly one note are large in number. (They are not listed here for C=0, since they do not enable smooth progressions. Of course, one

still can use them as part progressions when it is musically dsirable.)

Further, one might wonder whether one could define a harmonically meaningful and musically pleasing "smooth progression of 7th chords" by moving from 7th chord to 7th chord – analogously to the establishments of smooth progressions of minor and major triads in the Tonnetz. For example, one could consider the progression of 7th chords **CEGB → CEGBb → CEbGBb → CEbGbBb → CEbGbBbb.** Each chord in this progression differs by only one note from the previous chord, thus representing minimal change. More generally, it seems therefore feasible to include both triads and 7th chords in a large, smooth progressions.

The following table summarizes the sets of integer values of notes – for sets of cardinalities three, two and one -- that determine what notes must be shared with 3-note triad neighbors:

0	1	2	3	4	5	6	7	8	9	10	11	[0]
C	d	D	e	E	F	g	G	a	A	b	B	[C]
0			3			6						0
0			3						9			0
0						6			9			0
			3			6			9			
0			3									0
0						6						0
0									9			0

	3	6		
	3		9	
		6	9	
0				0
	3			
		6		
			9	

This table merely focuses on the integer values associated with the four notes of a (fully) diminished 7th chord, referring to its root note as 0. C=0 is only an example.

As far as "note and chord neighborhoods and degrees similarity" are concerned, one can use such complexes, commonly used in computational geometry, to determine sets of chords (in the same local complex neighborhood) sharing one note, two notes, three notes, four or more notes. Consequently, these sets are beneficial when determining smooth (or non-smooth) chord progressions from a start to a goal chord -- moving from chord to chord where large (or small) numbers of chord notes are kept (or not kept) in every chord-to-chord move in progressions.

Hierarchy. One could also consider exploring hierarchical concepts. For example, three obvious hierarchical levels of harmony are the level of the key, the level of the "main chords" in that key, and eventually the level of all notes in the key's scale. This is an example of such a possible three-level hierarchy: C maj → { I = CEG, ii = DFA, iii = EGB, IV = FAC, V = GBD, vi = ACE, vii = BDF } → { C, D, E, F, G, A,

B } is a hierarchy of three levels of C major, consisting of the key of C maj, the set of seven key-inherent triads, and the set of the key's individual notes. Is it possible to use "hierarchical thinking" when characterizing harmonic key and chord relationships and devising progressions using such harmonic hierarchies for mathematical and geometrical approaches?

Church modes. The concepts discussed here concern first and foremost the major (Ionian) and minor (Aeolian) scales, as defined by the numbers of semi-tone steps taken when moving from the root note of a scale to the same note 12 semi-tones higher. For completeness, all church modes and their defining sequences of 1- and 2-semi-tone steps are:

(1)	**IONIAN:**	**root+2+2+1+2+2+2+1**	**(CDEFGABC)**
(2)	Dorian :	root+2+1+2+2+2+1+2	(DEFGABCD)
(3)	Phrygian:	root+1+2+2+2+1+2+2	(EFGABCDE)
(4)	Lydian:	root+2+2+2+1+2+2+1	(FGABCDEF)
(5)	Mixolydian:	root+2+2+1+2+2+1+2	(GABCDEFG)
(6)	**AEOLIAN:**	**root+2+1+2+2+1+2+2**	**(ABCDEFGA)**
(7)	Locrian:	root+1+2+2+1+2+2+2	(BCDEFGAB)

Processing and searching in these discrete structures. When the described graphs, complexes or tessellations of notes, chords or keys/scales are established, one intends to use them as a basis for performing various kinds of relevant and musically interesting operations. For example, one could fix a note and "move around this note" by defining smooth progressions that all involve this not; or one could define a start and a goal chord (or a start key and a goal key) and determine the/a "shortest path" that connects start and

goal with a smooth progression of chords; or one could determine various kinds of smooth chord progressions that move “as continuously as possible, i.e., with minimal harmonic changes at every step” – moving between the fixed specified chords in an ordered sequence, using always the/a chord progression of minimal length when transitioning from one chord to the next in the desired sequence. This concept is like the concept of interpolating a given sequence of points by an interpolating curve. In summary, the discussed discrete geometrical representations graph structures can support many ways to determine harmonically and musically meaningful and pleasant “journeys in the complicated high-dimensional landscape of notes, chords and keys.” Since the finite sizes of the discrete structures considered are relatively small, the computational effort necessary to perform many of the searches for paths and progressions should generally be acceptable – even though general computational complexity of such searches in large graph structures can be prohibitive in settings where data set sizes are significantly larger.

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