Tone Lattices:
Concepts for Generalizing
the Tonnetz of Leonhard Euler

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Notes on the Structure of Musical Harmony
and Relationships with
Algebra, Combinatorics, Geometry, Graph Theory, Matrices, Modular Arithmetic, Number Theory, Circles of Notes and Chords,
the Tonnetz and Tone Lattices

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Sacramento, California
October 2026

“The supreme goal of all theory is to make the irreducible basic elements as simple and as few as possible without having to surrender the adequate representation of a single datum of experience."
— Albert Einstein

“You know, I couldn't do it. I couldn't reduce it to the freshman level. That means we really don't understand it.”
— Richard Feynman

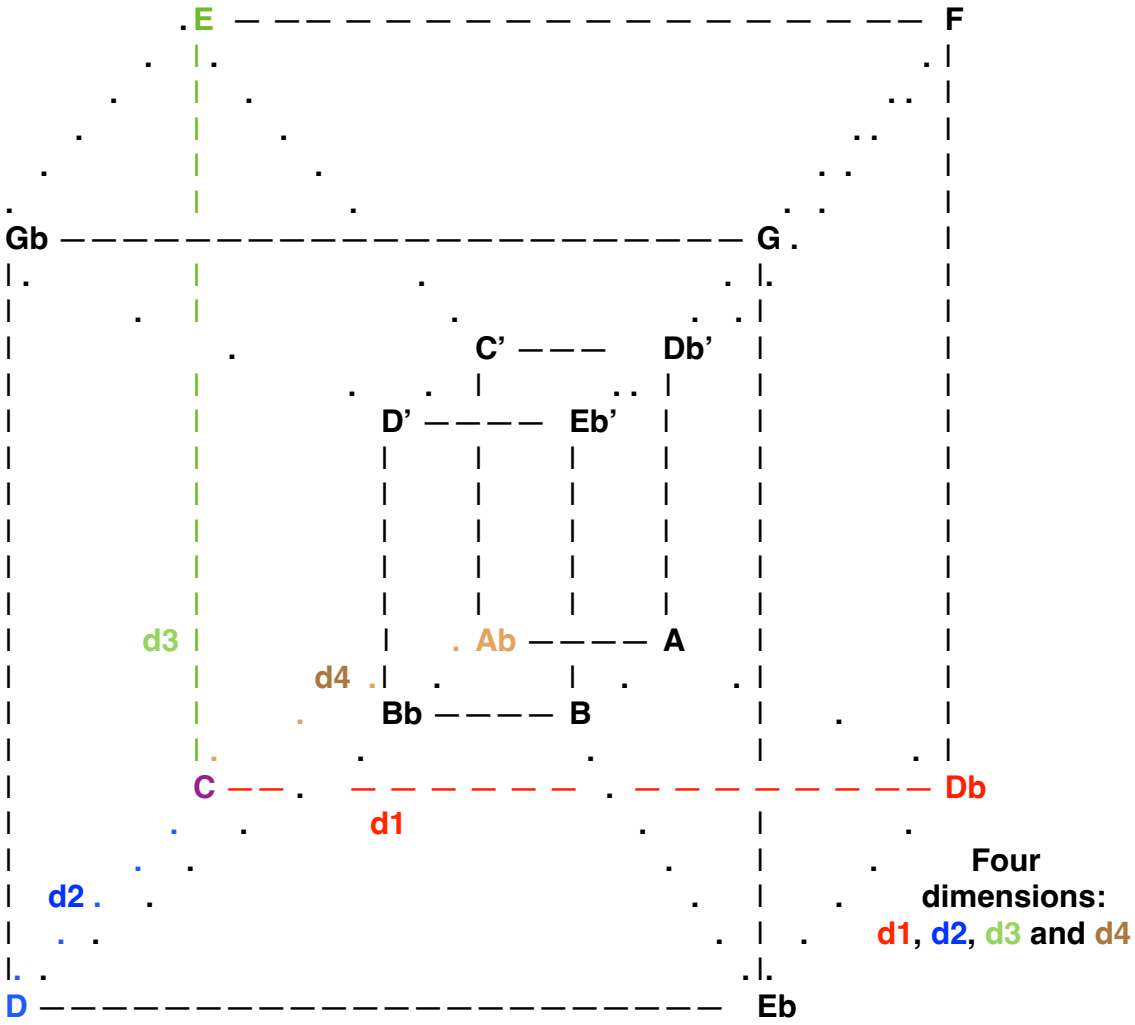
Introduction. These notes describe mathematical and geometrical concepts and ideas and their relationships to structures of harmony in music. The notes are not mathematically rigorous; they do not cover strict definitions or theorems; they merely provide a variety of mathematics-based approaches for possible ways to describe musical harmonic principles and their tendency to reflect certain underlying and governing rules that can be viewed via mathematical techniques. Further, the notes do not follow established music theory and its nomenclature for harmony. The notes present a collection of different ways to comprehend the structure of harmony by employing known methods that arise, e.g., in algebra, geometry, graph theory and combinatorics. Instead of using a highly abstract or general language for presentation, the notes often discuss facts and principles redundantly, using basic, colloquial language or specific simple examples. Illustrations and tables are used throughout to let readers “see” and develop their own approaches to establish a formal and broadly applicable theory, using sound mathematics and correct techniques from music theory. Concepts are discussed several times, often with simple and sometimes with complex sketches, in an attempt to explain the observed high-dimensional representations of harmonic structures and relationships. While the use of “somewhat inconsistent language” might seem undesirable, it serves the purpose of trying to describe complexity in a multitude of ways, including the use of colloquial language and illustrated simple examples while avoiding an unnecessarily abstract and general mathematical presentation. Several concepts and examples are repeatedly reviewed in the notes, usually for the purpose of describing important aspects in slightly different ways, serving a better understanding at the cost of employing redundant expositions.

“Mathematics and music, the most sharply contrasted fields of scientific activity which can be found, and yet related, supporting each other, as if to show forth the secret connection which ties together all the activities of our mind, and which leads us to surmise that the manifestations of the artist’s genius are but the unconscious expressions of a mysteriously acting rationality.”
—Hermann von Helmholtz

If these notes stimulate a person's thinking about meaningful ways to relate mathematics and music, then the notes will have served their purpose. The readers will find several errors and shortcomings, and it is hoped that they can forgive the author for using a style of language that is not in line with truly scientific or scholarly traditions routinely employed in music and mathematics.

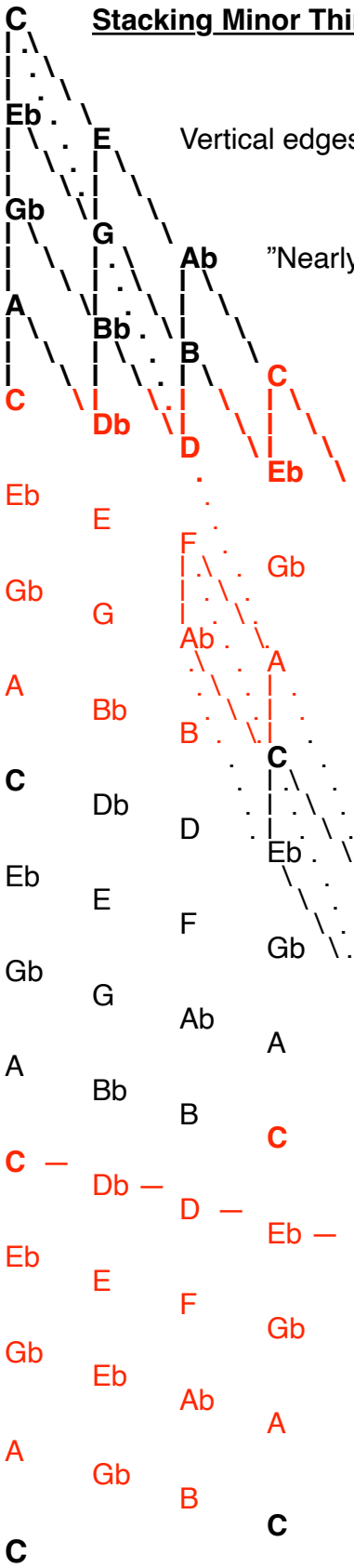
"Music is a science which should have definite rules; these rules should be drawn from an evident principle; and this principle cannot really be known to us without the aid of mathematics."
— Jean-Philippe Rameau

"Music is a higher revelation than all wisdom and philosophy."
— Ludwig van Beethoven



Cell of a tone lattice. Oblique-perspective projection of a 4-dimensional rectilinear grid cell (hyper-cube) into the plane, representing the 2^4=16 2-note chords C-C, …, C-C', … and C-Eb', with C', … and Eb' being next-octave notes.

Stacking Minor Thirds (Moving Down) & Major Thirds (Moving Down-right)



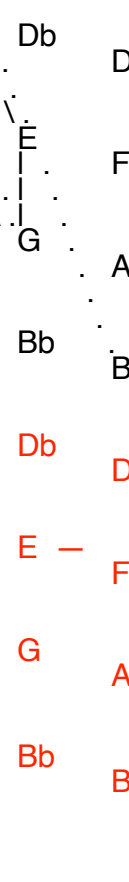
Vertical edges define minor thirds, diagonal edges define major thirds.

”Nearly horizontal” lines (C-Db-D-Eb-…) show the chromatic-scale.

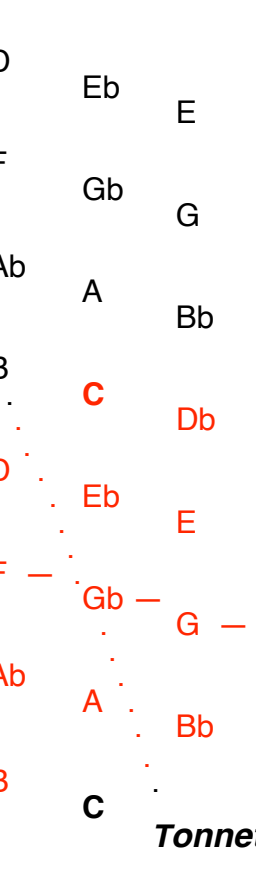
Edges shown as dotted lines show parts of circles of fifths.

Vertical lines define circles of four minor thirds.

Diagonals define circles of three major thirds.



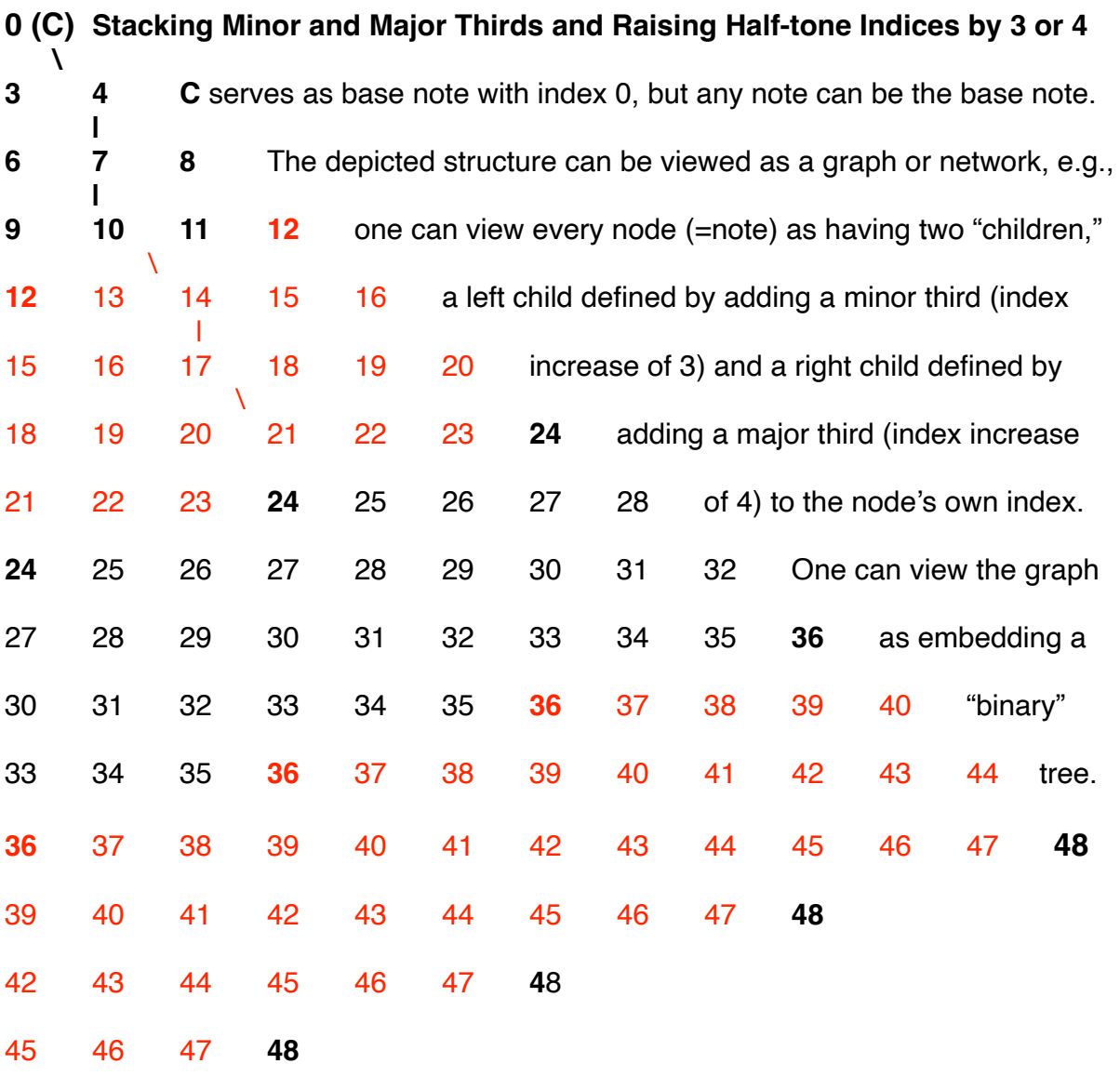
Six edges are connected to C: These edges define the chords (triads)



(Ab,C,Eb), (A,C,E), (F,Ab,C), (F,A,C), (C,Eb,G) and (C,E,G).



***Tonnetz* of Leonhard Euler, see [1]**



48 (C)

All possible “stacked chords” consisting of only minor and major thirds are paths in this structure. By moving down one adds a minor third, by moving down-right one adds a major third to a chord. One can “omit” a specific note(s) formally belonging to a chord consisting of minor and major thirds to simplify playing the chord, i.e., play the chord “incompletely.”

Examples: The 7th chord (C,E,G,Bb) has the associated index tuple (0,4,7,10).
The index tuple (0,4,7,10,14,17,21) is the 13th chord (C,E,G,Bb,D,F,A).
The path from C (0) to A (21) is shown.

Note. Slightly different notations are used to write chords. For example, the triad

consisting of the notes C, E and G, is written as (C,E,G) or C-E-G or CEG. Further, these notations assume that the notes are "linearly ordered" from left to right," i.e., the leftmost note is the lowest, …, and the rightmost is the highest.

Note. Leonhard Euler described the *Tonnetz* structure in 1739, see [1].

0 (C) Modulo-12 Arithmetic Using Cyclic, Periodic Half-tone Indices 0, …, 11

3 4 When placing the 12 notes/halftones of the chromatic scale periodically

6 7 8 on a circle, one can map every note/half-tone to an index in the

9 10 11 0 index set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}. The implied

0 1 2 3 4 tone-to-tone intervals from 0 to 0, from 0 to 1, …

3 4 5 6 7 8 and from 0 to 11 are: perfect unison (u);

6 7 8 9 10 11 **0** min and maj 2nd; min and maj 3rd;

9 10 11 **0** 1 2 3 4 perfect 4th; diminished 5th;

0 1 2 3 4 5 6 7 8 perfect 5th; min and

3 4 5 6 7 8 9 10 11 **0** maj 6th; min

6 7 8 9 10 11 **0** 1 2 3 4 and maj

9 10 11 **0** 1 2 3 4 5 6 7 8 7th

0 1 2 3 4 5 6 7 8 9 10 11 **0**

3 4 5 6 7 8 9 10 11 **0**

6 7 8 9 10 11 **0**

9 10 11 **0**

0 (C)

Note. To further simplify and shorten the notation for the possible tone-to-tone intervals, one can use the abbreviations 'p' for perfect, 'm' for minor and 'M' for major. The 13 intervals of one octave can then be written as p1, m2, M2, m3, M3, p4, d5, p5, m6, M6, m7, M7 and p8 (perfect 8th).

Using modulo-12 arithmetic allows one to view chord notes within a single 12-tone scale. Effectively, this cyclic (periodic) viewpoint makes possible the “translation” of a chord note(s) into the chosen 12-tone scale. While the diagram associates the index 0 with C, the index 0 can be associated with any note. Consequently, when “translating” a chord’s notes into a single 12-tone scale — thus, making the chord more “compact” — one obtains “clusters” of notes/tones, which also changes the “texture” of the original chord. Jazz, for example, uses these “clusters” extensively.

Example: The 9th chord (C, E, G, Bb, **D**) could be played as (C, **D**, E, G, Bb).

Periodic, Cyclic 12-tone Harmony and Matrices, Graphs and 12-dimensional Tori

The following 13-by-13 symmetric, periodic, cyclic matrix consists of rows and columns that are obtained by adding the numbers 0, 1, 2, …, 11, 0(=12) to the first row and first column containing the base value 0. These numbers are added in modulo-12 arithmetic, row-wise and column-wise. The numbers (indices) 0, 1, 2, …, 11 are also associated with the 12 half-tones of the 12-tone chromatic scale. A “circle” is understood as a path (sequence) of numbers in a specific row or column starting with a 0 and ending with a 0. For, example, the “+7 row” contains the half-tone indices 0, 7, 2, 9, 4, …, 5, 0, defining the half-tones of the circle of fifths (0 being the index of the base note C), i.e., the circle consisting of the notes C, G, D, A, …, F, C. The matrix highlights the rows and columns containing the circle of fifths (moving forward as C, G, D, … in the “+7” row and the “+7” column or moving backward as C, F, A#, … in the “+5” row and the “+5” column). One can understand the “+5” row and the “+5” column as a circle of fourth (moving forward).

+0	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+0
0	0	0	0	0	0	0	0	0	0	0	0	0 | +0
0	1	2	3	4	5	6	7	8	9	10	11	0 | +1
0	2	4	6	8	10	0	2	4	6	8	10	0 | +2
0	3	6	9	0	3	6	9	0	3	6	9	0 | +3
0	4	8	0	4	8	0	4	8	0	4	8	0 | +4
0	5	10	3	8	1	6	11	4	9	2	7	0 | +5
0	6	0	6	0	6	0	6	0	6	0	6	0 | +6
0	7	2	9	4	11	6	1	8	3	10	5	0 | +7
0	8	4	0	8	4	0	8	4	0	8	4	0 | +8
0	9	6	3	0	9	6	3	0	9	6	3	0 | +9
0	10	8	6	4	2	0	10	8	6	4	2	0 |+10
0	11	10	9	8	7	6	5	4	3	2	1	0 |+11
0	0	0	0	0	0	0	0	0	0	0	0	0 | +0

Note. In modulo-12 arithmetic, adding 1 corresponds to subtracting 11 etc., i.e.,

+0 == -0, +1 == -11, +2 == -10, +3 == -9, +4 == -8, +5 == -7,
+6 == -6, +7 == -5, +8 == -4, +9 == -3, +10 == -2, +11 == -1.
The "==" notation is being "abused" here, and one should properly interpret it as: Value to the left of "==" corresponds to value to the right of "==". One can think of a clock with 12 hour marks: A specific mark/hour can be reached by either moving forward (clockwise) by k hours from the 12 o'clock mark or by moving backward (counter-clockwise) by (12-k) hours from the 12 o'clock mark.

Note. One can view this matrix also as a graph or network, e.g., as a structure where each entry/node has 12 children — since it is possible to add 0, 1, … or 11 to a node's value. It is a challenge to come up with a visualization of such a complex structure that is not cluttered.

Note. Further, since every row and column of the matrix contains at least one cycle (circle) from 0 to 0, the geometry of such a "multi-periodic structure" can also be viewed as a high-dimensional (12-dimensional), multi-periodic "hyper-torus." (Computer networks often use 2-, 3- or even higher-dimensional toroidal node connectivity.)

The following matrix shows the specific half-tones that are defined when using the base tone C. Here, the matrix only uses the '#' symbol/operator — "for the black piano keys."

+0	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+0	
C	C	C	C	C	C	C	C	C	C	C	C	C	+0
C	C#	D	D#	E	F	F#	G	G#	A	A#	B	C	+1
C	D	E	F#	G#	A#	C	D	E	F#	G#	A#	C	+2
C	D#	F#	A	C	D#	F#	A	C	D#	F#	A	C	+3
C	E	G#	C	E	G#	C	E	G#	C	E	G#	C	+4
C	F	A#	D#	G#	C#	F#	B	E	A	D	G	C	+5
C	F#	C	F#	C	F#	C	F#	C	F#	C	F#	C	+6
C	G	D	A	E	B	F#	C#	G#	D#	A#	F	C	+7
C	G#	E	C	G#	E	C	G#	E	C	G#	E	C	+8
C	A	F#	D#	C	A	F#	D#	C	A	F#	D#	C	+9
C	A#	G#	F#	E	D	C	A#	G#	F#	E	D	C	+10
C	B	A#	A	G#	G	F#	F	E	D#	D	C#	C	+11
C	C	C	C	C	C	C	C	C	C	C	C	C	+0

The following matrix uses both the '#' and the 'b' notation for the representation of the sequences of half-tones appearing in specific rows and columns. When reading a row from left to right or a column from top to bottom, one can more easily recognize circles of half-tones when written in the "standard way." The "+5" row shows the familiar circle sequence C, F, Bb, Eb, … (reading the row from left to right), obtained when reading the standard circle of fifths backward from the 12 o'clock (= C) mark as well.

+0	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+0
C	C	C	C	C	C	C	C	C	C	C	C	C I +0
C	C#	D	Eb	E	F	F#	G	Ab	A	A#	B	C I +1
C	D	E	Gb	G#	Bb	C	D	E	Gb	G#	A#	C I +2
C	Eb	Gb	A	C	Eb	Gb	A	C	Eb	Gb	A	C I +3
C	E	G#	C	E	G#/Ab	C	E	G#/Ab	C	E	G#	C I +4
C	F	Bb	Eb	G#/Ab	Db	Gb	B	E	A	D	G	C I +5
C	F#	C	Gb	C	Gb	C	F#	C	Gb	C	F#	C I +6
C	G	D	A	E	B	F#	C#	G#/Ab	D#/Eb	A#	F	C I +7
C	G#/Ab	E	C	G#/Ab	E	C	G#/Ab	E	C	Ab/G#	E	C I +8
C	A	Gb	Eb	C	A	Gb	D#/Eb	C	A	Gb	Eb	C I +9
C	A#	G#	Gb	E	D	C	A#	G#/Ab	Gb	E	D	C I+10
C	B	A#	A	G#	G	F#	F	E	Eb	D	C#	C I+11
C	C	C	C	C	C	C	C	C	C	C	C	C I +0

One can apply “thinning” to the matrix of modulo-12 indices by listing only one circle per row, leading to the simplified and more abstract matrix shown next.

0	.	.	.	.	.	.	.	.	.	.	.	0 I +0
0	1	2	3	4	5	6	7	8	9	10	11	0 I +1
0	.	2	.	4	.	6	.	8	.	10	.	0 I +2
0	.	.	3	.	.	6	.	.	9	.	.	0 I +3
0	.	.	.	4	.	.	.	8	.	.	.	0 I +4
0	5	10	3	8	1	6	11	4	9	2	7	0 I +5
0	.	.	.	.	.	6	.	.	.	.	.	0 I +6
0	7	2	9	4	11	6	1	8	3	10	5	0 I +7
0	.	.	.	8	.	.	.	4	.	.	.	0 I +8
0	.	.	9	.	.	6	.	.	3	.	.	0 I +9
0	.	10	.	8	.	6	.	4	.	2	.	0 I+10
0	11	10	9	8	7	6	5	4	3	2	1	0 I+11
0	.	.	.	.	.	.	.	.	.	.	.	0 I +0

Matrix with rows consisting of circles (from 0 to 0, in each row) obtained by adding 0(=12), 1, 2, 3, …, 11.

This matrix is “point-symmetric” relative to its center, having 6 at its column-midpoint. While this representation is rather abstract, one can recognize that the index sequence 0, 7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5, 0 (from left to right) in the “+7” row defines the circle of fifths. Assuming that 0 represents C, for example, the sequence of half-tones is

C, G, D, E, B, F#, C#, G#, D#, A#, F(E#), C(B#). The "+5" row represents the circle of fourths, having indices 0, 5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7, 0, implying the half-tone sequence C, F, Bb, Eb, Ab, Db, Gb, B, E, A, D, G, C. To be formally correct, one writes B as Cb, E as Fb, A as Bbb, D as Ebb and G as Abb. The "+3" row is also of interest, as it shows the index values for the circle of minor thirds, i.e., it lists the index sequence **0, 3, 6, 9, 0**, which defines the half-tone sequence C, Eb, Gb, A, C, again assuming that 0 stands for the C. Since the matrix is characterized by point symmetry, the note indices shown in matrix positions (I,J) and (J,I) are the same. The three circles described are highlighted in the matrix.

One can simplify the matrix above further via "contraction." The left-most column of the matrix shown above consists only of the index value 0. Thus, one can compress the left-most column to a single entry (node) with value 0. One can now move forward (upward on a scale) by adding 0, 1, … or 11 half-tones, indicated in the representation below by the paths defined by the horizontal and vertical line segments emanating from 0 and ending at 0, 1, 2, … or 11. Analogously, the right-most column of the matrix shown above can be compressed to one 0. The circle of minor thirds (+3), the circle of fourths (+5) and the circle of fifths (+7) are highlighted again. It is not obvious how one could effectively use such a highly abstract representation for the construction or the analysis of complex k-note chords.

```
   I— 0   —  —  —  —  —  —  —  —  —  — -I    +0
   I— 1   2  3  4  5  6  7  8  9 10 11 -I    +1
   I— 2   4  6  8 10  —  —  —  —  —  — -I    +2
   I— 3   6  9  —  —  —  —  —  —  —  — -I    +3
   I— 4   8  —  —  —  —  —  —  —  —  — -I    +4
   I— 5  10  3  8  1  6 11  4  9  2  7 -I    +5
0—— 6   —  —  —  —  —  —  —  —  —  — -I—0  +6
   I— 7   2  9  4 11  6  1  8  3 10  5 -I    +7
   I— 8   4  —  —  —  —  —  —  —  —  — -I    +8
   I— 9   6  3  —  —  —  —  —  —  —  — -I    +9
   I— 10  8  6  4  2  —  —  —  —  —  — -I    +10
   I— 11 10  9  8  7  6  5  4  3  2  1 -I    +11
   I— 0   —  —  —  —  —  —  —  —  —  — -I    +0
```

Contracted row-based representation of circles, from 0 to 0, using modulo-12 arithmetic.

Construction of n-dimensional Tori with n-directional Lattice Structures

One can use the concept of a hyper-cube (or hyper-cuboid) — the high-dimensional generalization of the 3-dimensional cube — combined with imposing a lattice structure, where "points" (nodes) are placed uniformly in all n axis directions of an n-dimensional hyper-cube. In the case of the 2-dimensional tonnetz, one can view this it as follows:

The circles of minor thirds and circles of major thirds are used to define a 2-dimensional cube in the plane with a superimposed lattice structure consisting of “points” (nodes), together with half-tone notes of the underlying circles. The lattice is constructed in a periodic way, i.e., the boundaries in each dimension (or axis direction) repeat cyclically. This regular lattice structure can be thought of as defining the 2-dimensional periodic “parametric domain” of a standard 2-dimensional torus embedded in 3-dimensional space as a surface — i.e., a surface that is bi-periodic. This viewpoint and construction can be adopted for the definition of lattice structures for hyper-cube of dimension 0, 1, 2, … or 12. For example, a standard 3-dimensional cube can be used as domain of a lattice structure that represents circles of minor thirds in the first dimension (first-axis direction), circles of fourths in the second dimension (second-axis direction) and circles of fifths in the third dimension (third-axis direction). The resulting cube would have identical “left and right faces,” identical “front and back faces” and identical “bottom and top faces,” reflecting the periodicity of the underlying circles.

In order to understand such a construction, one should first consider low-dimensional examples before moving to the general high-dimensional setting. For the construction of a simple 2-dimensional bi-cyclic, bi-periodic lattice, we use the circles of minor thirds perfect fifths. First, one must establish the 12 equivalent (shifted) circles of minor thirds and the 12 equivalent circles of perfect fifths — by using as base indices (base notes) the indices 0, 1, 2, …, 11 with corresponding half-tones C, C#/Db, D, D#/Eb, E, F, … defining the beginning.

Note. From now on, we use exclusively the ‘b’ operator, avoiding the ‘#’ operator, simply to keep the presentation shorter. The reader will be able to properly interpret notes as lowered or raised notes based on context.

The 12 shifted circles of minor thirds (with five notes associated with each circle) are

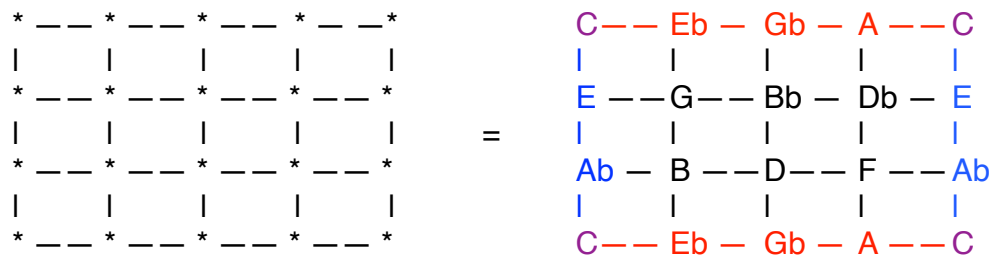
(C, Eb, Gb, A, C),	(Eb, Gb, A, C, Eb),	(Gb, A, C, Eb, Gb),	(A, C, Eb, Gb, A),
(Db, E, G, Bb, Db),	(E, G, Bb, Db, E),	(G, Bb, Db, E, G),	(Bb, Db, E, G, Bb),
(D, F, Ab, B, D),	(F, Ab, B, D, F),	(Ab, B, D, F, Ab),	(B, D, F, Ab, B).

In fact, each row lists the four positions (root position and three inversions) of one of the three possible fully diminished 7th chords. We use the term “shifting” as a alternative
term to refer to the process of choosing the base notes of these circles as C, Db or D.
The 12 shifted circles of major thirds (with 4 notes associated with each circle) are

(C, E, Ab, C),	(E, Ab, C, E),	(Ab, C, E, Ab),
(Db, F, A, Db),	(F, A, Db, F),	(A, Db, F, A),
(D, Gb, Bb, D),	(Gb, Bb, D, Gb),	(Bb, D, Gb, Bb),
(Eb, G, B, Eb),	(G, B, Eb, G),	(B, Eb, G, B).

Here, the respective base notes are C, Db, E and F, the listed circles represent root positions and two inversions for each base note.

We can now define a 2-dimensional cube (a rectangle) with a 2-dimensional lattice structure to establish the domain of the standard tonnetz used for representing the relationships between circles of minor and major thirds and implied minor and major triads. First, we arrange circles of minor thirds in rows (first dimension) and circles of major thirds in columns (second dimension). Second, we combine the 1-dimensional circles of minor thirds with the circles of major thirds in a rectilinear grid fashion, to define the desired 2-directional, 2-dimensional lattice structure that is bi-cyclic, bi-periodic. The total resolution of the final lattice structure is (4+1) x (3+1) = 20.

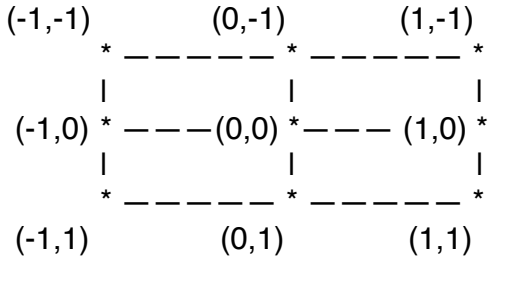


Rectilinear grid (left) consisting of points (vertices) and orthogonal edges connecting points. The annotated grid (right) shows note names at the points that represent minor-third movements (from left to right) and major-third movements (from top to bottom). Thus, one must keep in mind that notes have different “note interval distances” in the two directions.

This lattice structure is indeed periodic: The left and right columns are identical, and the bottom and top rows are identical. The notes associated with the points (nodes) reflect this periodicity. This structure captures the circles of (perfect) fifths in one of the two diagonal directions, e.g., C-G-D-A, and circles of minor seconds in the other direction, e.g., C-Db-D-Eb. The shown lattice structure is the tonnetz structure that can be mapped parametrically and cyclically onto the surface of the standard torus.

Note. The lattice structure shown in the figure above (left) is also depicting the geometrical *rhombic (ortho-rhombic) atomic structure of crystals* where a crystal lattice exhibits only 90-degree angles where lattice edges intersect. One can understand such a rhombic structure as a rectilinear grid.

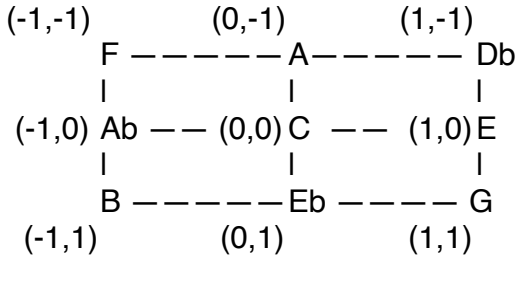
Usually, one employs a multi-dimensional index, based on n-dimensional tuples, to enumerate the points/vertices of an n-dimensional rectilinear grid. For example, one can use the following 2-dimensional tuples to index points of a 2-dimensional rectilinear grid, shown in the figure for a neighborhood around a center point/vertex:



Lattice of **two dimensions:**
Dimension 1: Horizontal edges
Dimension 2: Vertical edges
Four oriented axis directions:
Left to right: (1,0)
Right to left: (-1,0)
Top to bottom: (0,1)
Bottom to top: (0,-1)

Four diagonal directions (direction vectors): (1,1), (-1,-1), (1,-1), (-1,1)

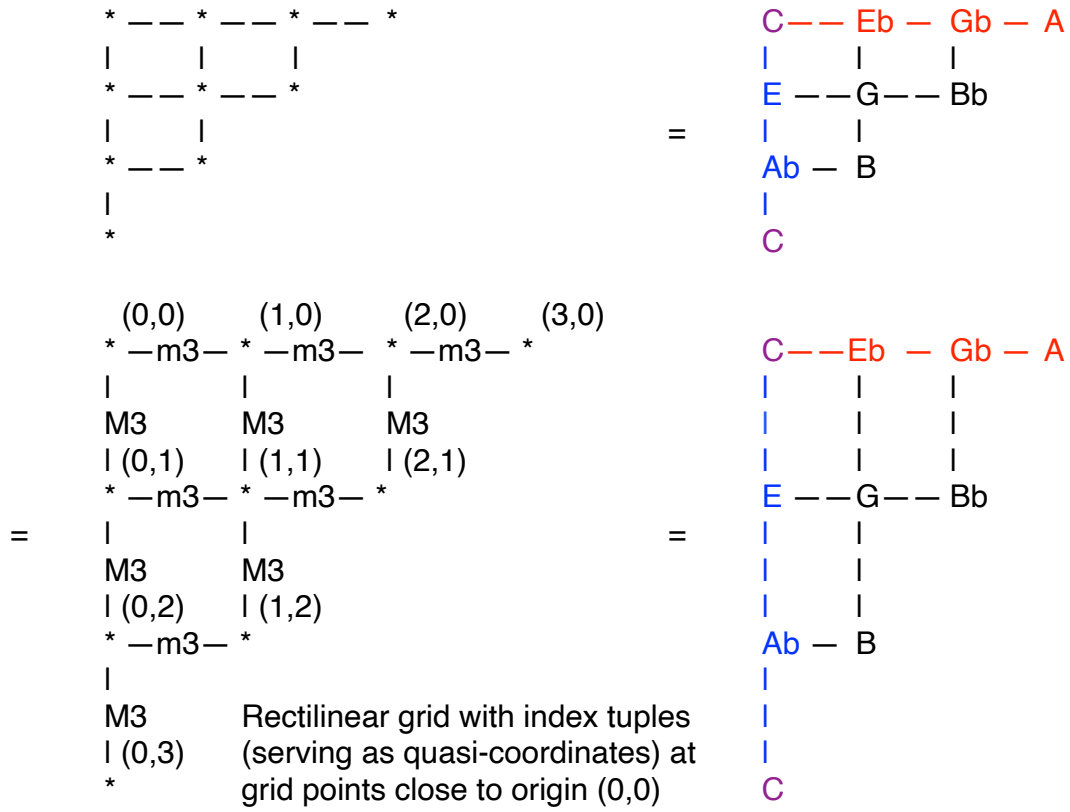
When annotating the points with note names one sees the tonnetz:



Local view of **tonnetz around C**.
Triads including C are shown:
(F,Ab,C), (F,A,C), (Ab,C,E),
(A,C,E), (C,Eb,G) and (C,E,G).
The (1,0) direction represents
major thirds, the (0,1) direction
represents minor thirds, the
“implied diagonal” directions
(1,1) and (1,-1) represent perfect fifths and minor seconds, respectively.
The tonnetz can this be understood as a **2-dimensional cube lattice structure where every 2x2 cell encode**s four “forward” (+) movements: **unison (u), m3, M3 and p5 movements**.

Generally, a 0- / 1- / 2- / 3- / … / n-dimensional rectilinear grid or rhombic lattice consists of cells of dimension 0 / 1 / 2 / 3 / … / n having 2^0=0 / 2^1=2 / 2^2=4, 2^3=8 / 2^n corner points.

It is also possible to understand the tonnetz as a means for representing stacking of notes to define 2-note, 3-note, 4-note, …, k-note chords. When moving from left to right in direction (1,0) and when moving from top to bottom in direction (0,1), one obtains the results depicted in the detailed figure below, where stacking minor 3rds and major 3rds is indicated via m3 and M3 edge annotations in the rectilinear grid.



One sees that this grid stencil of 10 notes represents all possible 1-, 2-, 3- and 4-note chords on the base note C. Combinatorially, 2^0=1, 2^1=2, 2^2=4 and and 2^3=8 1-, 2-, 3- and 4-note chords exist. One can think of these chords as chords defined by moving in sequences of possible m3 and M3 intervals (steps) from note to note. We provide a list of these chords:

2^0: — —> C (unison = u);
2^1: m3M3 —> C-Eb (min 3rd), C-E (maj 3rd);
2^2: m3m3, m3M3, M3m3, M3M3 —> C-Eb-Gb (Cdim), C-Eb-G (Cmin), C-E-G (Cmaj), C-E-Ab (Caug);

2^3: m3m3m3, m3m3M3 —> C-Eb-Gb-A (Cfulldim7), C-Eb-Gb-Bb (Chalfdim7);
m3M3m3, m3M3M3 —> C-Eb-G-Bb (Cm7), C-Eb-G-B (CmM7=Cminmaj7);
M3m3m3, M3m3M3 —> C-E-G-B (Cmaj7), C-E-G-Bb (C7);
M3M3m3, M3M3M3 —> C-E-Ab-Bb (Caug7), C-E-Ab-B (Cmaj7#5).

One can think of the progression of dimensions as adding one dimension at a time. In the context of note intervals and rectilinear grids, one can visualize a progression by using only forward (+) movements from the base note C that serves as local origin:

2^0 = 1
1 interval (u)

C
*
0

2^1 = 2
2 2-note intervals (u, m3)

C m3 Eb
* ——————— *
0 1

2^2 = 4
4 3-note intervals (u, m3, M3, p5)

C m3 Eb
(0,0) * ——————— * (1,0)
| . |
M3 | . | Standard tonnetz
| p5 . |
(0,1) * ——————— * (1,1)
E G

2^3 = 8
8 4-note intervals (u, m3, M3, m2, p5, M3, p4, m6)

C m3 Eb
(0,0,0) * ——————— * (1,0,0)
| . |
M3 | . |
| p5 . |
(0,1,0) *——————— * (1,1,0)
E G

Standard tonnetz (bottom layer) extracted into third dimension, representing 8 intervals on base (root) C: C-C, C-Eb, C-E, C-Db, C-G, C-E, C-F and C-Ab (C-E appearing twice).

. Bottom layer with
. note annotations
.
.
.
Extraction into .
3rd dimension .
.

Db m3 E
(0,0,1) * ——————— * (1,0,1)
| . |
M3 | . |
| p5 . |
(0,1,1) * ——————— * (1,1,1)
F Ab

See Banchoff [2]

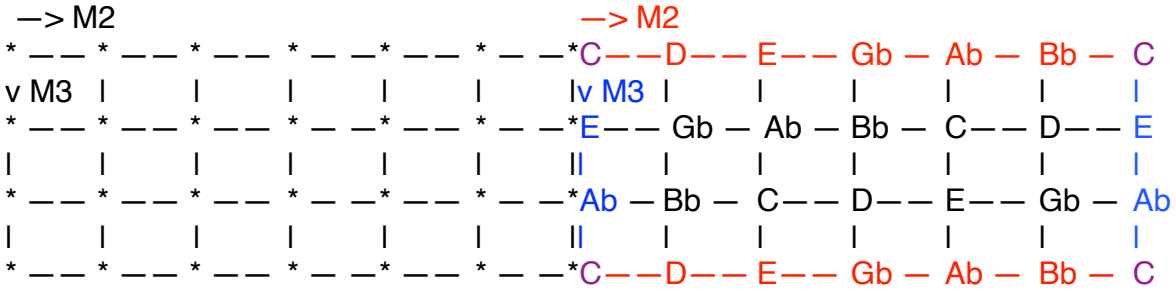
Top-layer notes: m2 interval above bottom-layer notes

In summary, the tonnetz based purely on m3- and M3-stacking in its (1,0)- and (0,1)-directions, represents all possible ways to stack any combination of minor and major 3rds by moving in its inherent rectilinear gris structure in a “zigzag” fashion, moving either by an m3 or M3 at every note when moving forward.

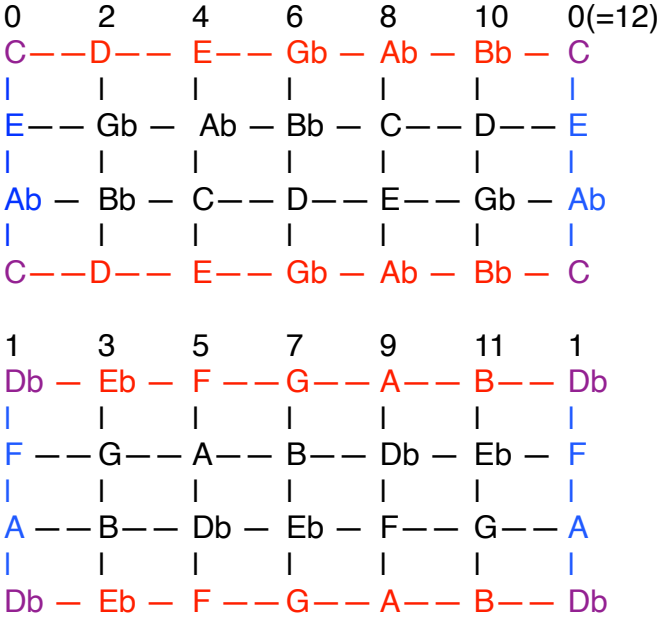
Next, we consider the construction of a lattice structure for the circles of major seconds and the circles of major thirds. We must use the 12 shifted circles of major seconds (with seven notes associated with each circle); the circles are

(C, D, E, Gb, Ab, Bb, C), (D, E, Gb, Ab, Bb, C, D), (E, Gb, Ab, Bb, C, D, E),
(Gb, Ab, Bb, C, D, E, Gb), (Ab, Bb, C, D, E, Gb, Ab), (Bb, C, D, E, Gb, Ab, Bb),
(Db, Eb, F, G, A, B, Db), (Eb, F, G, A, B, Db, Eb), (F, G, A, B, Db, Eb, F),
(G, A, B, Db, Eb, F, G), (A, B, Db, Eb, F, G, A), (B, Db, Eb, F, G, A, B).

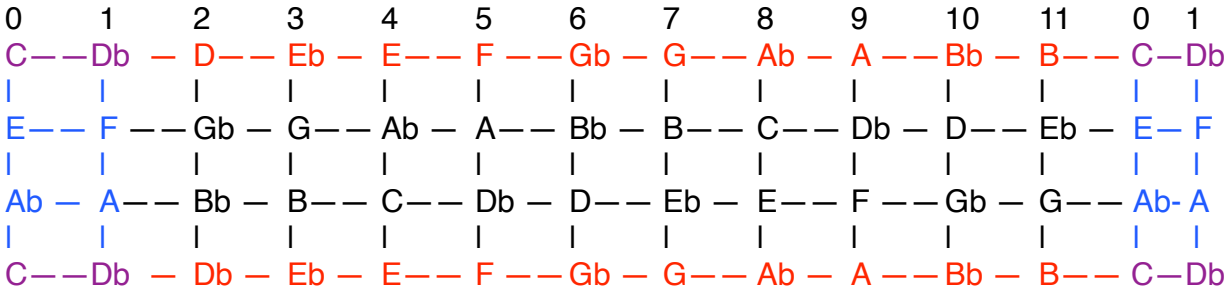
The two possible base notes are C and Db. We arrange the circles of major seconds in rows and the circles of major thirds in columns. This next figure shows the resulting lattice structure.



Here, the purely point-based lattice structure (left) is shown next to the annotated structure (right), where points have associated note names. This structure is again periodic in both grid directions. In this case, the resolution is (6+1) x (3+1) = 28.
In this structure, one diagonal direction shows tritone sequences, e.g., C-Gb-C-Gb, and circles of major seconds in the other diagonal direction, e.g., Gb-Ab-Bb-C. One sees that the notes Db, Eb, F, G, A, and B do not appear in the annotated structure. This fact is a consequence of the prime factors of 2 and 4: 2 = 1 x 2 and 4 = 2 x 2, which explains the missing note names: Multiples of 2 are even numbers, and the uneven numbers — the indices of the missing notes — are not, cannot be covered, i.e., Db=1, Eb=3, F=5, G=7, A=9 and B=11. In order to complete the required lattice, one must add/insert additional rows/columns. The next figure shows two identical lattice structures for (i) the six notes with even indices and (ii) the six notes with uneven indices that are missing in the figure above. Note indices are included in the figure above the notes in the top row.



For example, one can use an “insertion” of the structure representing the notes with uneven indices into the structure representing the notes with even indices, i.e., the four rows of the second structure can be “inserted” into the corresponding rows of the first structure as illustrated in the next figure.



One can think of this combined structure as follows: The structure is “double-periodic” in the first dimension (row direction), i.e., the two left-most columns are both repeated in identical form as right-most columns. The top and bottom rows are identical as well, reflecting periodic behavior in the second dimension. This “superposition” of the two lattices with even and uneven note indices has resolution (2 x (6+1)) x (3+1) = 56. Thus, in every row circles of major seconds are defined by note sequences obtained by skipping every other note, e.g., consider the circles C-D-E-Gb-… or Db-Eb-F-G-… Each column represents a circle of major thirds. It is important to note that circles of major thirds are repeated multiple times in the columns, a consequence of combining the two 2-dimensional lattice structures as described here. Ignoring the two right-most columns in the figure above, one sees that each circle of major thirds is captured exactly three times, e.g., the circle C-E-Ab-C is also represented in inverted forms E-Ab-C-E (column with index 4) and Ab-C-E-Ab (column with index 8). Further, one of the diagonal directions defines circles of perfect fourths, e.g., consider the sequence C-F-Bb-Eb being part of a circle of perfect fourths; and the other diagonal direction captures circles of major sixths, e.g., consider the sequence C-A-Gb-Eb being part of a circle of major sixths.

We now discuss the 3-dimensional case, leading to a tri-periodic or tri-cyclic network structure based on a rectilinear grid using 3-dimensional cubes as grid cells. When considering the simplest grid consisting of merely a single cell, the cell is a cube with periodic face pairs, i.e., “left face = right face,” “front face = back face” and “bottom face = top face.” As an example, we use the circles of minor thirds, circles of major thirds and circles of perfect fifths to define a 3-directional lattice structure. Using only the ‘b’ operator for consistency of notation and simplification, the list of 12 shifted circles of perfect fifths (13 notes per circle) consists of these circles:

(C, G, D, A, E, B, Gb, Db, Ab, Eb, Bb, F, C),
(Db, Ab, Eb, Bb, F, C, G, D, A, E, B, Gb, Db),
(D, A, E, B, Gb, Db, Ab, Eb, Bb, F, C, G, D),
(Eb, Bb, F, C, G, D, A, E, B, Gb, Db, Ab, Eb),

(E, B, Gb, Db, Ab, Eb, Bb, F, C, G, D, A, E),
(F, C, G, D, A, E, B, Gb, Db, Ab, Eb, Bb, F),
(Gb, Db, Ab, Eb, Bb, F, C, G, D, A, E, B, Gb),
(G, D, A, E, B, Gb, Db, Ab, Eb, Bb, F, C, G),
(Ab, Eb, Bb, F, C, G, D, A, E, B, Gb, Db, Ab),
(A, E, B, Gb, Db, Ab, Eb, Bb, F, C, G, D, A),
(Bb, F, C, G, D, A, E, B, Gb, Db, Ab, Eb, Bb),
(B, Gb, Db, Ab, Eb, Bb, F, C, G, D, A, E, B).

We have already constructed a 2-dimensional lattice for the circles of minor and major thirds. In order to add the circle of perfect fifths, one can utilize the third dimension by “stacking” 12 2-dimensional lattice structures serving as “layers” of the cube lattice, moving form the bottom face to the top face. For example, one can stack the 2-dimensional lattices as illustrated in the figure below — by simply moving up by one half-tone per lattice-point when moving upward for the construction of layers 0, 1, 2, …, 12(=0). The ‘+’ operator is used in the figure to represent the stacking of layers.

Layer 0 (Bottom Face)

C — — Eb — Gb — A — — C
| | | | |
E — — G — — Bb — Db — E
| | | | |
Ab — B — — D — — F — — Ab
| | | | |
C — — Eb — Gb — A — — C

+

Layer 1

Db — E — G — Bb — Db
| | | | |
F — — Ab — B — — D — — F
| | | | |
A — — C — — Eb — Gb — A
| | | | |
Db — E — — G — — Bb — Db

+

Layer 2

D — — F — — Ab — B — — D
| | | | |
Gb — A — — C — — Eb — Gb
| | | | |
Bb — Db — E — — G — — Bb
| | | | |
D — — F — — Ab — B — — D

+

Layer 3

Eb — Gb — A — — C — — Eb
| | | | |
G — — Bb — Db — E — — G
| | | | |
B — — D — — F — — Ab — B
| | | | |
Eb — Gb — A — — C — — Eb

+ Layer 4 + Layer 5 + Layer 6 + Layer 7 + Layer 8 + Layer 9

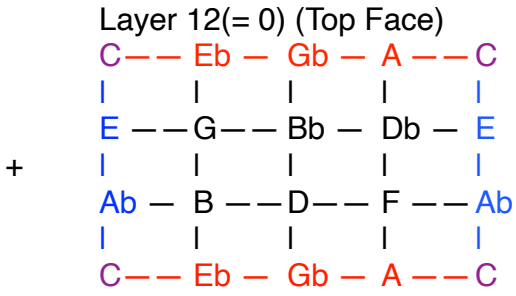
Layer 10

Bb — Db — E — — G — — Bb
| | | | |
D — — F — — Ab — B — — D
| | | | |
Gb — A — — C — — Eb — Gb
| | | | |
Bb — Eb — E — — G — — Bb

+

Layer 11

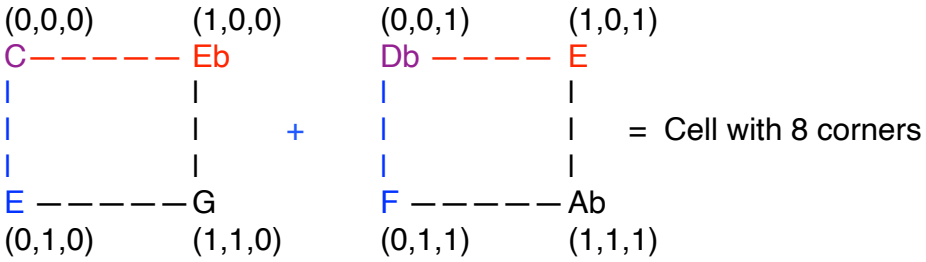
B — — D — — F — — Ab — B
| | | | |
Eb — Gb — A — — C — — Eb
| | | | |
G — — Bb — Db — E — — G
| | | | |
B — — D — — F — — Ab — B



One sees that each layer has periodic boundaries and that each layer uses annotations that include every half-tone of the chromatic scale, with different arrangements of the note names. Further, the four corners of the set of 13 layers define the half-note sequence C, Db, D, Eb, …, Bb, B, C, as expected. Furthermore, every lattice position of the 13 2-dimensional lattice layers has a specific "column in the third dimension" when moving from bottom to top face, defining circles of minor seconds. For example, the point in layer-0 with note G has the notes Ab, A, Bb, …, F, Gb, G in the subsequent layers 1, 2, …, 10, 11, 12(=0).

As it is difficult to visualize this 3-dimensional annotated lattice structure via a projection of the embedding cube, the figure simply shows the individual 2-dimensional layer lattices. The total resolution is (3+1)x(4+1)x(12+1)=260.

Having constructed such a 3-dimensional lattice, one can explore whether the edges of the lattice and paths ("piecewise linear paths") in it represent specific chords, relevant note sequences, circles or other characteristics of harmony that are of interest. For example, we consider the upper-left 2 x 2 parts of layers 0 and 1 (of a 3-dimensional lattice structure) and stack them, defining the simplest single-cell, single-cube grid, shown below.



In this 3-dimensional case, there exist eight directions one can consider for moving forward (+) or backward (-) in the single cell:

- The "zero direction vector" (0,0,0) defines unison (u). The corresponding interval is C-C.
- The three grid directions (1,0,0), (0,1,0) and (0,0,1) define minor thirds, major thirds and minor seconds, respectively.

The corresponding intervals are C-Eb, C-E and C-Db.

— The three cell-face directions (1,1,0), (1,0,1) and (0,1,1) represent perfect fifths, major thirds and perfect fourths, respectively. The corresponding intervals are C-G, C-E and C-F.

— The cell-interior diagonal direction (1,1,1) defines minor sixths. The corresponding interval is C-Ab.

This list of represented intervals includes the intervals arising in the standard tonnetz, i.e., minor thirds, major thirds and perfect fifths. Thus, the described 3-dimensional lattice structure contains the 2-dimensional tonnetz as a "by-product." Of course, the 3-dimensional lattice structure also represents "complementary intervals" to the ones listed above. The eight complementary directions (moving in opposite directions) are based on the directions (-0,-0,-0), (-1,0,0), (0,-1,0), (0,0,-1), (-1,-1, 0), (-1,0,-1), (0,-1,-1) and (-1,-1,-1). Complementarity is explained by the underlying modulo-12 arithmetic of the chromatic scale. This is the list of the complementary intervals that arise:

Unison (u)	—> Octave	(0 —> 12: 0+12 = 12=0),
Minor third	—> Major sixth	(3 —> 9: 3+9 = 12=0),
Major third	—> Minor sixth	(4 —> 8: 4+8 = 12=0),
Minor second	—> Major seventh	(1 —> 11: 1+11 = 12=0),
Perfect fifth	—> Perfect fourth	(7 —> 5: 7+5 = 12 = 0).

Thus, the only intervals not represented in the 3-dimensional lattice structure are major second and minor seventh.

The above figure of one cell with the eight corners C, Db, Eb, E, F, G, Ab also defines 4-note chords when using a paths consisting of four notes for moving from C to G and using the positive directions provided above. Formally possible paths — defining 4-note chords — from C to Ab are C-Eb-E-Ab, C-Eb-G-Ab, C-E-F-Ab, C-E-G-Ab, C-Db-E-Ab and C-Db-F-Ab (using only forward directions). These paths are the "zigzag" paths that can take 90-degree turns at every point/note in the rectilinear grid when moving to a subsequent point/note. One also can keep in mind that the distinct seven different notes associated with the cube's corners can be viewed simply as a set of seven notes from which one would choose four notes; there exist 7! / (4! x (7-4)!) = 7! / (4! x 3!) = 35 4-note chords (with mutually different chord notes). One can determine whether some of these formally possible chords are harmonically relevant. For example, when considering such a purely combinatorial method for constructing chords, the chord Db-E-Ab-C, a seventh chord, is also represented. The sequence of forward (+) and backward (-) directions/direction vectors producing this seventh chord, with base note Db, is the sequence of directions (1,0,0), (0,1,0) and (-1,-1,-1), i.e., one must include the negative direction (-1,-1,-1) in the sequence.

We consider the local neighborhood of 3x3x3 notes around the center note C with associated local grid origin triple (0,0,0). We consider three layers: layer -1, layer 0 and layer 1 (using a layer numbering that reflects layer-stacking in the third dimension, i.e., when stacking layers from the cube's bottom face to the cube's top face in the third dimension. The illustration below shows local lattice "coordinate triples," relative to the center point (origin) triple (0,0,0) with associated note C.

Layer -1
(-1,-1,-1) (0,-1,-1) (1,-1,-1)
(-1,0,-1) (0,0,-1) (1,0,-1)
(-1,1,-1) (0,1,-1) (1,1,-1)

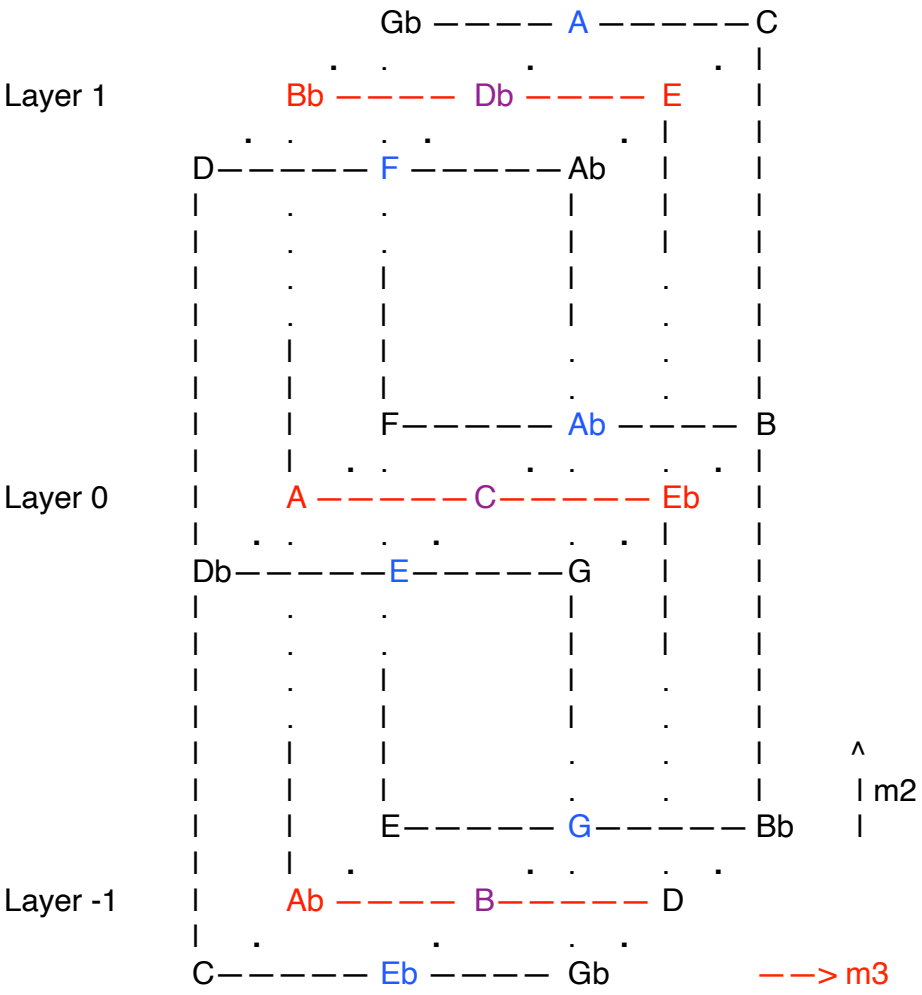
E — G — Bb
| | |
Ab — B — D
| | |
C — Eb — Gb

Layer 0
(-1,-1,0) (0,-1,0) (1,-1,0)
(-1,0,0) (0,0,0) (1,0,0)
(-1,1,0) (0,1,0) (1,1,0)

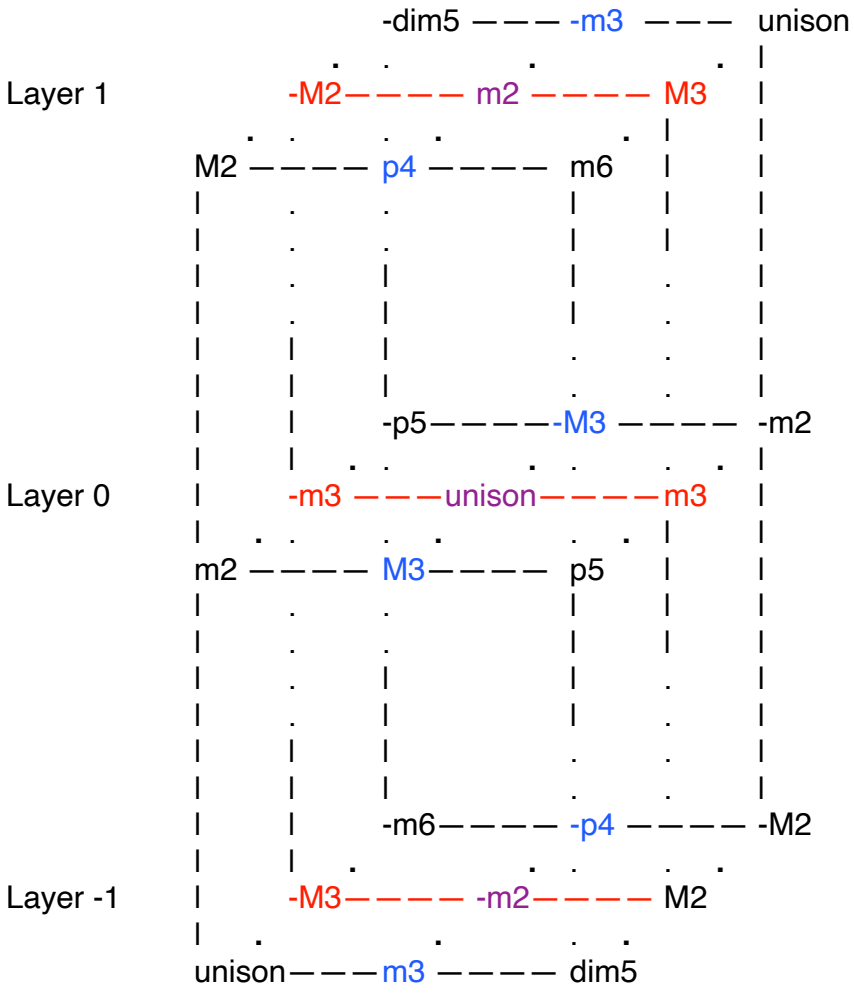
F — Ab — B
M3 | | | |
v A — C — Eb
| | |
Db — E — G
—> m2

Layer 1
(-1,-1,1) (0,-1,1) (1,-1,1)
(-1,0,1) (0,0,1) (1,0,1)
(-1,1,1) (0,1,1) (1,1,1)

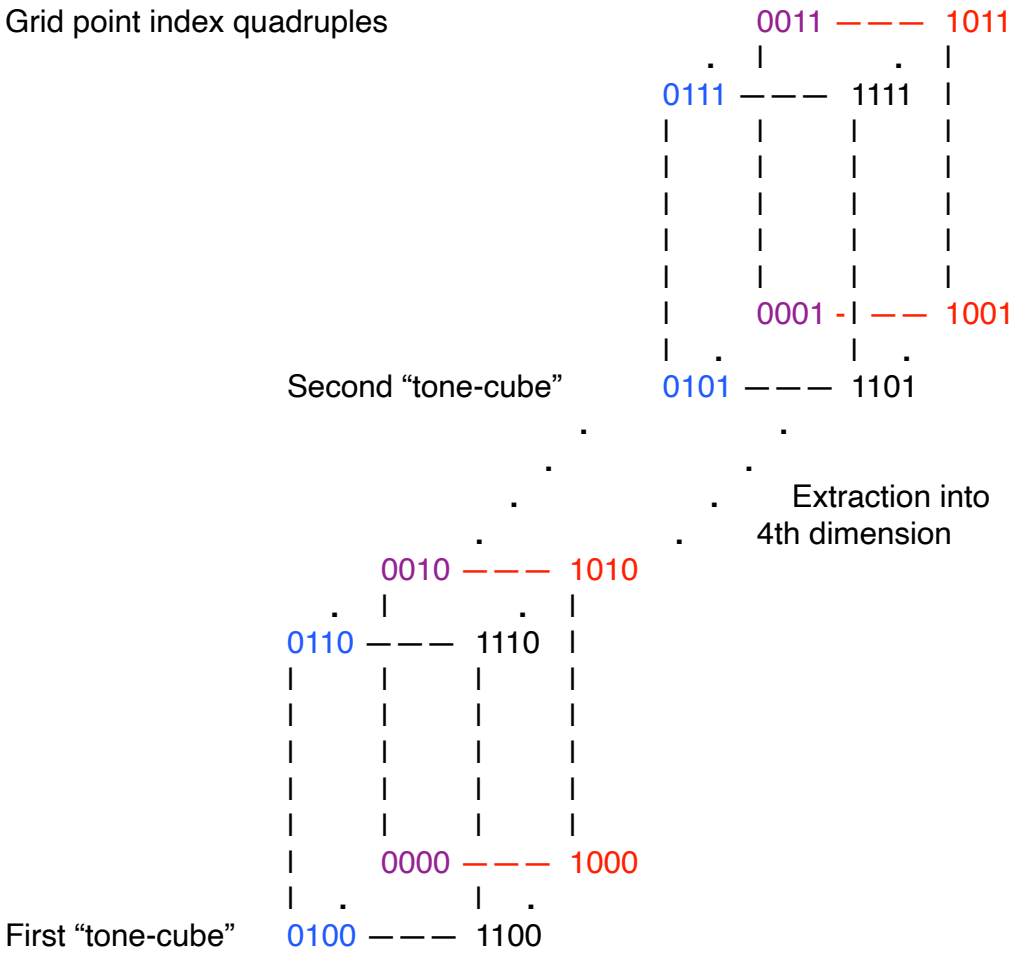
Gb — A — C
| | |
Bb — Db — E
| | |
D — F — Ab



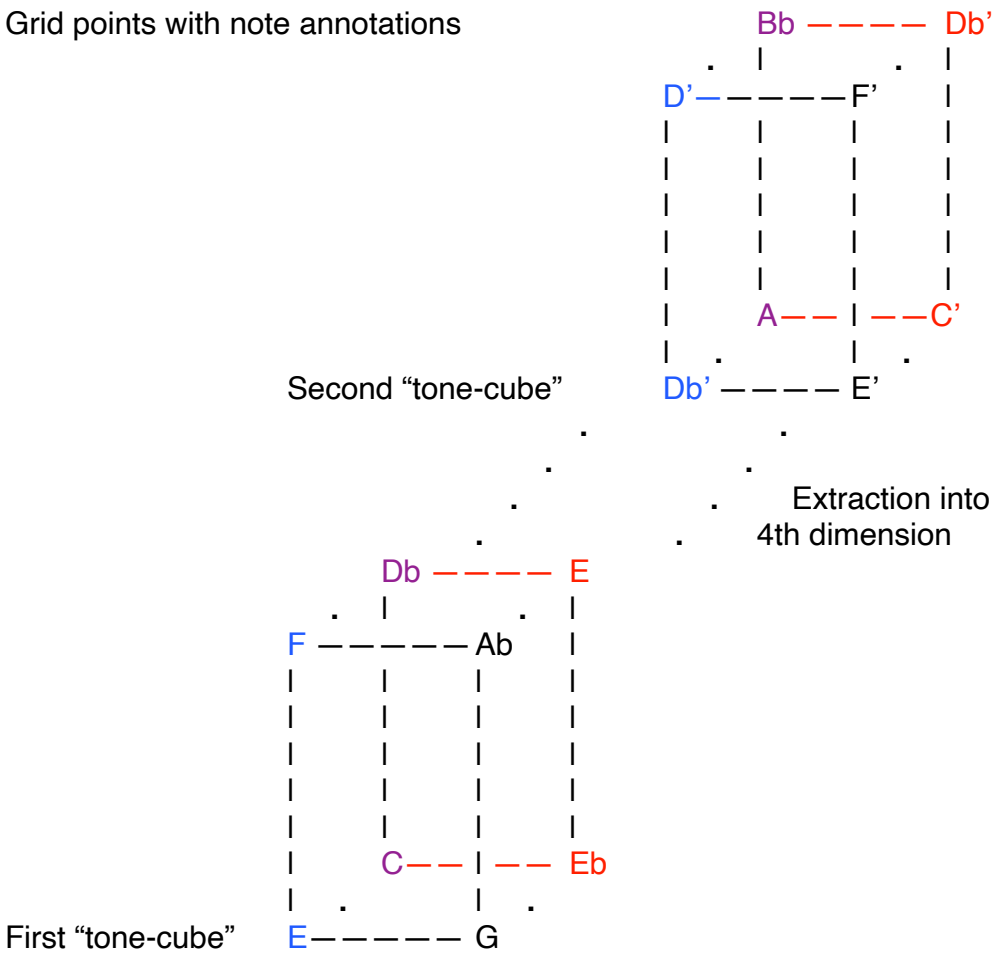
Oblique projection of a 3-dimensional tone lattice consisting of 27 points (notes) and 8 cubes defining a local rectilinear grid. Corresponding intervals relative to center note C — “moving from C to …” — making visible the underlying point-symmetric behavior, relative to the center point (C), which includes these Intervals: 3 unisons (u), 2 pairs (m2, -m2), 2 pairs (M2, -M2), 2 pairs (m3, -m3), 2 pairs (M3, -M3), 1 pair (p4, -p4), 1 pair (dim5, -dim5), 1 pair (p5, -p5) and 1 pair (m6, -m6).

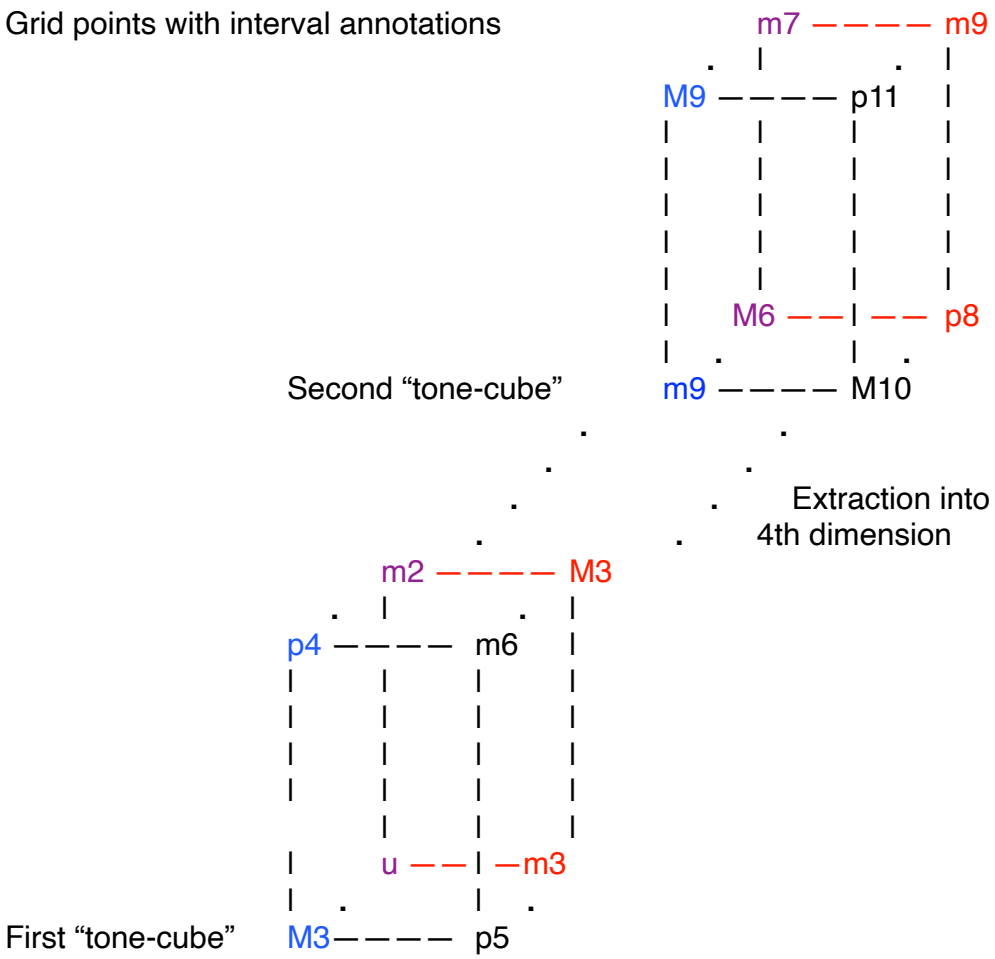


Next, we consider the 4-dimensional setting. First, we merely explore a single rectilinear grid cell, i.e., a hyper-cube with 16 corners.



Extraction of a cube into fourth dimension by adding a major sixth (M6) to every note, generating a local grid stencil, tone lattice, with 2^4 = 16 points, with C’ etc. indicating notes in the subsequent octave. This visualization uses a translation of an oblique projection of a cube in the plane.


Grid points with note annotations
Bb
Db'
D'
F'
A
C'
Db'
E'
Second "tone-cube"
Extraction into
4th dimension
Db
E
F
Ab
C
Eb
First "tone-cube"
E
G



It is possible to identify sequences of note intervals associated with the forward grid directions/direction vectors of rectilinear grids, starting at their local origins (0), (0,0), (0,0,0) etc. that permit direct movements, direct “jump,” to all of the notes within an octave above the base note (and beyond). The following table summarizes the underlying combinatorial considerations. The table shows that a 4-dimensional hyper-cube suffices to cover all the 2^4 = 16 mutually different note intervals, defining the intervals from unison (u) to m10, i.e., from interval C-C to interval C-Eb’ (C’, C’’, Db’, Db’’, etc. being notes in subsequent octaves). The abbreviation ‘d’ is used for “dimension.” The table uses the concept of “adding” dimensions one-by-one (first column).

Add dim d	Number of grid Points	Implied powers of 2	Sums	Intervals	Notes
0	2^0 = 1	0	0+0 = 0	u+u=u	C
1	2^1 = 2	0, 1	0+0 = 0 0+1 = 1	u u+m2=m2	C Db

2	2^2 = 4	0, 1, 2	0+0 = 0	u	C
			0+1 = 1	m2	Db
			0+2 = 2	u+M2=M2	D
			1+2 = 3	m2+M3=m3	Eb
3	2^3 = 8	0, 1, 2, 4	0+0 = 0	u	C
			0+1 = 1	m2	Db
			0+2 = 2	M2	D
			1+2 = 3	m3	Eb
			0+4 = 4	M3	E
			1+4 = 5	p4	F
			2+4 = 6	d5	Gb
			1+2+4 = 7	p5	G
4	2^4 = 16	0, 1, 2, 4, 8	0+0 = 0	u	C
			0+1 = 1	m2	Db
			0+2 = 2	M2	D
			1+2 = 3	m3	Eb
			0+4 = 4	M3	E
			1+4 = 5	p4	F
			2+4 = 6	d5	Gb
			1+2+4 = 7	p5	G
			0+8 = 8	m6	Ab
			1+8 = 9	M6	A
			2+8 = 10	m7	Bb
			1+2+8 =11	M7	B
			4+8 = 12	p8	C’
			1+4+8 = 13	m9	Db’
			2+4+8 =14	M9	D’
			1+2+4+8 = 15	m10	Eb’
5	2^5 = 32	0, 1, 2, 4, 8, 16	0+0 = 0	u	C
			0+1 = 1	m2	Db
			0+2 = 2	M2	D
			1+2 = 3	m3	Eb
			0+4 = 4	M3	E
			1+4 = 5	p4	F
			2+4 = 6	d5	Gb
			1+2+4 = 7	p5	G
			0+8 = 8	m6	Ab
			1+8 = 9	M6	A
			2+8 = 10	m7	Bb
			1+2+8 =11	M7	B
			4+8 = 12	p8	C’
			1+4+8 = 13	m9	Db’
			2+4+8 =14	M9	D’
			1+2+4+8 = 15	m10	Eb’

0+16 = 16	M10	E’
1+16 = 17	p11	F’
2+16 = 18	d12	Gb’
1+2+16 = 19	p12	G’
4+16 = 20	m13	Ab’
1+4+16 = 21	M13	A’
2+4+16 = 22	m14	Bb’
1+2+4+16 = 23	M14	B’
8+16 = 24	p15	C”
1+8+16 = 25	m16	Db”
2+8+16 = 26	M16	D”
1+2+8+16 = 27	m17	Eb”
4+8+16 = 28	M17	E”
1+4+8+16 = 29	p18	F”
2+4+8+16 = 30	d19	Gb”
1+2+4+8+16 = 31	p19	G”

The information captured in this table can be presented in a more compact form by avoiding data duplication (redundant, repeated rows), shown in the next table:

Add dim d	Number of grid points	Implied powers of 2	Sums	Intervals	Notes
0	2^0 = 1	0	0+0 = 0	u	C
1	2^1 = 2	0, 1	0+1 = 1	m2	Db
2	2^2 = 4	0, 1, 2	0+2 = 2	M2	D
			1+2 = 3	m3	Eb
3	2^3 = 8	0, 1, 2, 4	0+4 = 4	M3	E
			1+4 = 5	p4	F
			2+4 = 6	d5	Gb
			1+2+4 = 7	p5	G
4	2^4 = 16	0, 1, 2, 4, 8	0+8 = 8	m6	Ab
			1+8 = 9	M6	A
			2+8 = 10	m7	Bb
			1+2+8 =11	M7	B
			4+8 = 12	p8	C’
			1+4+8 = 13	m9	Db’
			2+4+8 =14	M9	D’
			1+2+4+8 = 15	m10	Eb’

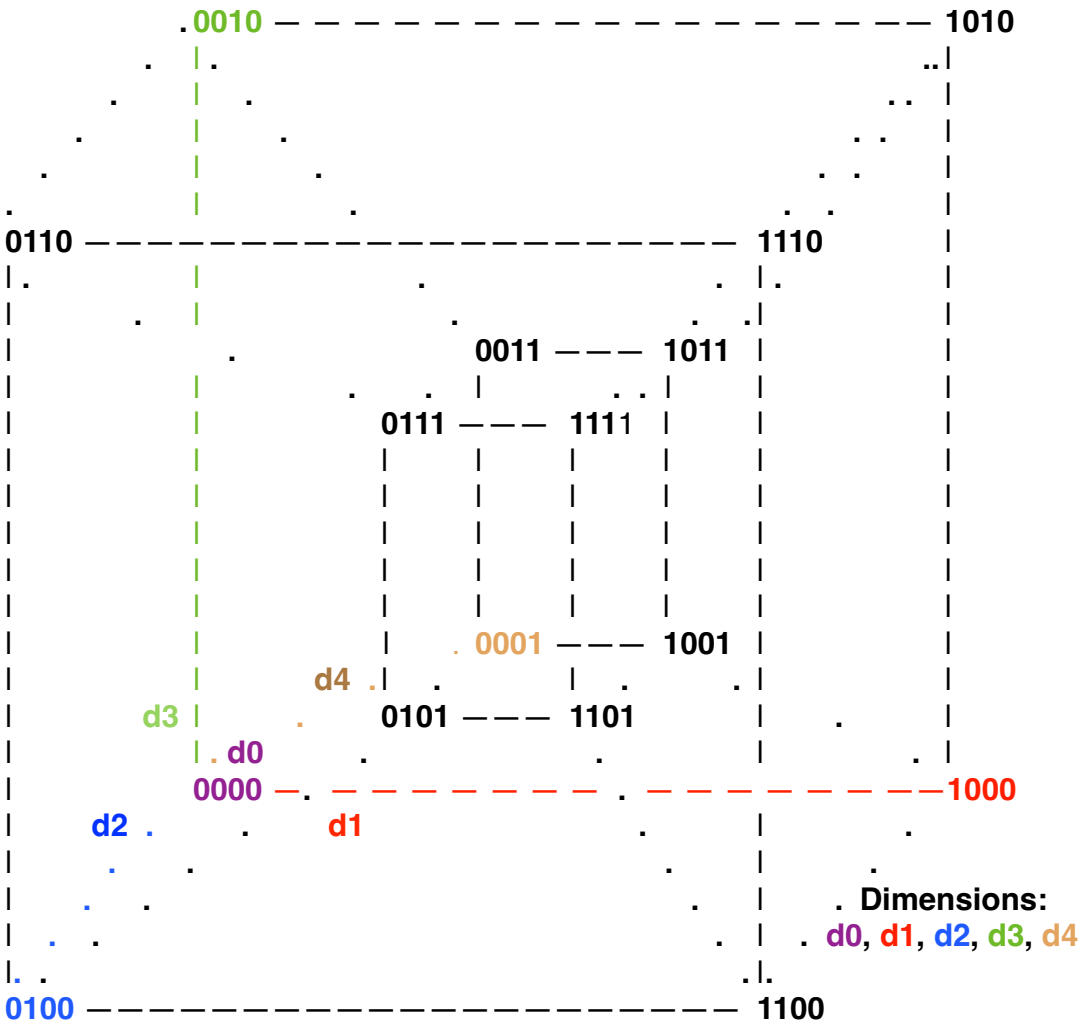
5	2^5 = 32	0, 1, 2, 4, 8, 16	0+16 = 16	M10	E’
			1+16 = 17	p11	F’
			2+16 = 18	d12	Gb’
			1+2+16 = 19	p12	G’
			4+16 = 20	m13	Ab’
			1+4+16 = 21	M13	A’
			2+4+16 = 22	m14	Bb’
			1+2+4+16 = 23	M14	B’
			8+16 = 24	p15	C”
			1+8+16 = 25	m16	Db”
			2+8+16 = 26	M16	D”
			1+2+8+16 = 27	m17	Eb”
			4+8+16 = 28	M17	E”
			1+4+8+16 = 29	p18	F”
			2+4+8+16 = 30	d19	Gb”
			1+2+4+8+16 = 31	p19	G”

	2^3 =8	2^2 =4	2^1 =2	2^0 =1	Sum dec	Sum in binary	Implied interval	Next note on base C
Multipliers	0	0	0	0	0	0000	u	C
0 or 1 used	0	0	0	1	1	0001	m2	Db
to multiply	0	0	1	0	2	0010	M2	D
the specific	0	0	1	1	3	0011	m3	Eb
powers of 2	0	1	0	0	4	0100	M3	E
shown in the	0	1	0	1	5	0101	p4	F
associated	0	1	1	0	6	0110	d5	Gb
columns at	0	1	1	1	7	0111	p5	G
the top of	1	0	0	0	8	1000	m6	Ab
the table	1	0	0	1	9	1001	M6	A
	1	0	1	0	10	1010	m7	Bb
	1	0	1	1	11	1011	M7	B
	1	1	0	0	12	1100	p8	C’
	1	1	0	1	13	1101	m9	Db’
	1	1	1	0	14	1110	M9	D’
	1	1	1	1	15	1111	m10	Eb’

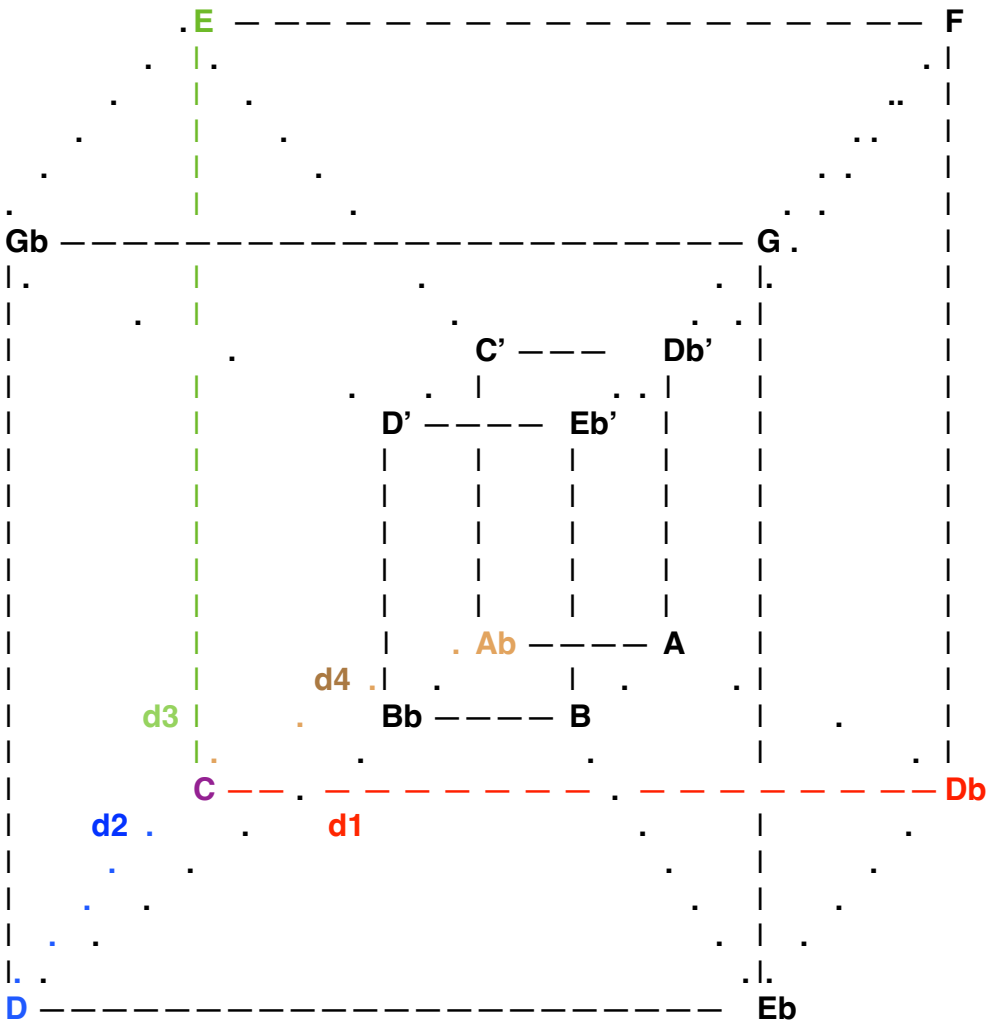
Table showing relationships between quadruples being the indices of the “prototype” 16-point grid cell of 4-dimensional rectilinear grid, with integers 0, …, 16 represented in decimal and binary notations and as powers of 2. This table explains why it is possible to use a single 4-dimensional local rectilinear cell grid cell to represent all 16 note intervals from, from u to m10, producing the implied linearly ordered notes from C to Eb’, serving as note annotations of the cell’s 16 corner points. The figures below show projections of a 4-dimensional cube into the plane. The “Sum in binary”

column provides the binary representations of the decimal numbers (“Sum dec”) shown in the same row, i.e., it provides the 4-bit number representations that have the bit associated with the smallest power of 2 (2^0) as their rightmost bit and the bit associated with the largest power of 2 (2^3) as leftmost bit.

Note. It is important to keep in mind the difference between “left-to-right” and “right-to-left” notations of numbers, multi-indices and the coordinate tuples of points in multi-dimensional space. Specifically, in “right-to-left” notation, the “most significant digit” (“most significant bit” in binary representation) of a number’s representation in some underlying basis is the left-most non-zero digit. Examples are these decimal number <—> binary number pairs: 0 <—> 0, 1 <—> 1, 2 <—> 10, 3 <—> 11, 4 <—> 100, …, 7 <—> 111. In contrast, multi-indices and point coordinate tuples in a multi-dimensional space must be read from left to right, since, for example, a point with positional vector (c1, c2, c3) represents the point reached from the origin by the (basis vector sum c1*b1 + c2*b2 + c3*b3, with b1, b2 and b3 being the ordered underlying basis vectors — e.g., the orthogonal basis vectors b1 = (1,0,0), b2 = (0,1,0) and b3 = (0,0,1). The indexing of grid points in a rectilinear grid in 3-dimensional space, for example, employs the same convention: A multi-index tuple (i1, i2, i3, …) uses index component i1 for the first dimension, index component i2 for the second dimension, index component i3 for the third dimension, etc. This distinction is relevant here as these different notations and their proper contextual interpretation must be kept in mind when seeing multi-indices of rectilinear grid points, coordinates of grid points and number representations in decimal or binary form. Generally, the context makes clear, whether one must understand a notation as a “right-to-left” or “left-to-right” notation.

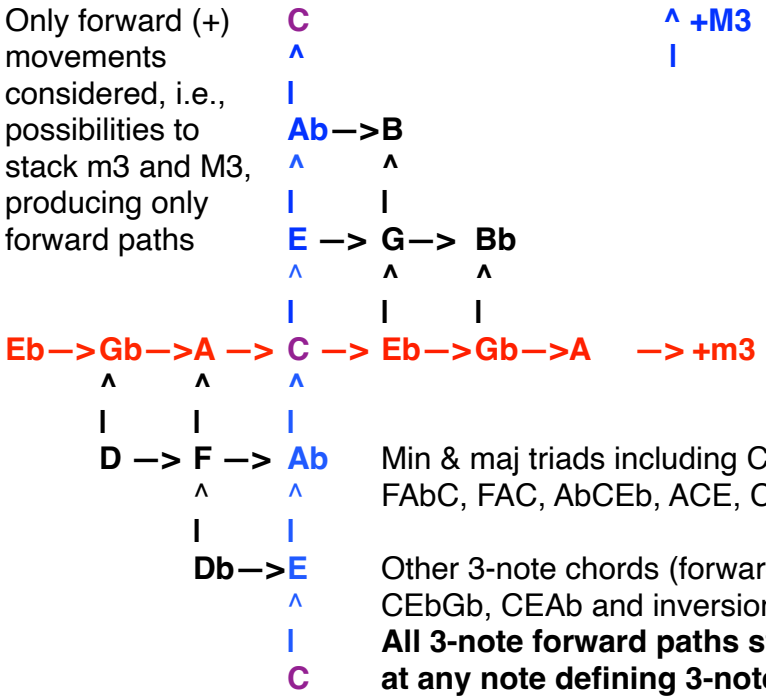


Visualization of 16-point grid stencil via a “combined oblique-perspective projection” of a 4-dimensional hyper-cube with dimensions **d0**, d1, d2, d3 and d4. Grid point index quadruples are written as 0000, …, 1111 for to simplify the notation. The grid is a 4-directional rectilinear grid with the four directions/direction vectors (1,0,0,0), (0,1,0,0), (0,0,1,0) and (0,0,0,1).



Using specific note names as point annotations of one hyper-cube grid cell.

A relevant question that arises is the following: The tonnetz elegantly defines the six triads (minor and major) of a note associated with a point, by considering the six edges emanating from the note's grid point, i.e, the edges with directions m3, -m3, M3, -M3 and the implied diagonal directions p5 and -p5. Does the presented local rectilinear grid point stencil of a 4-dimensional grid also define important chords including a note selected in the grid by considering edges emanating from the note's grid point in various directions? In order to answer this question we review and consider the 2- and 3-dimensional settings where the local neighborhood of a base note C is explored. We are interested in determining which 3- and 4-note chords can be identified in the sketched neighborhood of the standard tonnetz, here understood as a rectilinear grid.



4-note chords starting at any note (using m3m3m3, …, M3M3M3) are
Cmaj7 = CEGB, C7 = CEGBb,
Cm7 = CEbGBb,
Chalfdim7 = CEbGbBb, Cfulldim7 = CEbGbA (and its inversions)
Cminmaj7 = CEbGB, CEAb[C] = Caug (and its inversions), …
= all 4-note forward paths starting at any note defining 4-note chords.

One can think of these chords as chords obtained by defining polylines using grid edges, i.e., “paths” in a rectilinear grid stencil, where one can define any forward sequence of +m3 and +M3 movements starting at any grid point/note and ending at a note reached after moving by 2 or 3 thirds, using +m2 or +M3 steps. Here, all notes of the chromatic scale appear in the local grid stencil.

We now consider the 3-dimensional case. We consider the example of two rectilinear grid cells sharing one common cell corner grid point with base note C. The following figure below shows the projections of two 3-dimensional grid cells, cubes, that share one common cube corner grid point for the base note C. One can keep in mind these “equivalences” when thinking of the grid points as points with index triples, as points with note annotations or as points with note interval annotations. For the two cubes shown, the 12 “equivalences” (‘=‘) are

000 = C = u, 100 = Db = m2, 010 = D = M2, 110 = Eb = m3,
001 = E = M3, 101 = F = p4, 011 = Gb = d5, 111 = G = p5,
-0-0-0 = C = -u, -100 = B = -m2, 0-10 = Bb = -M2, -1-10 = A = -m3,
00-1 = Ab = -M3, -10-1 = G = -p4, 0-1-1 = Gb = -d5, -1-1-1 = F = -P5.

The six 3-note chords visualized by the standard tonnetz are also represented

in the figure below, defined by forming polylines connecting grid points with corresponding note annotations, using the rectilinear edges and also forward--diagonals of cell faces (= sums of two forward grid edges); they are

Fmin = FAbC, Fmaj = FAC, Abmaj = AbCEb, Amin = ACE,
Cmin = CEbG, Cmaj = CEG.

In addition, one can identify polylines in this local grid point stencil that define other meaningful chords including C, including incomplete chords with missing notes written as "[…]" or chord inversions. Examples are

Cdim = CEbGb, Cmaj7 = CE[G]B, Abaug = AbCE, ACEb = Adim,
Csus2 = CDG, Csus4 = CFG, C7 = CEGBb, Cm7 = CEbGBb,
Chalfdim = CEbGbBb, Cfulldim7 = CEbGbA, …

In the shown stencil, the note C is surrounded by 10 different notes lying either in the octave of which C is the lowest note or in the octave of which C is the highest note. One can formally construct 45 3-note chords and 120 4-note chords consisting of C and 2 or 3 of its 10 "surrounding" neighbor notes, respectively, since 10C2 = 45 and 10C3 = 120 ("nCk" abbreviating "n choose k").

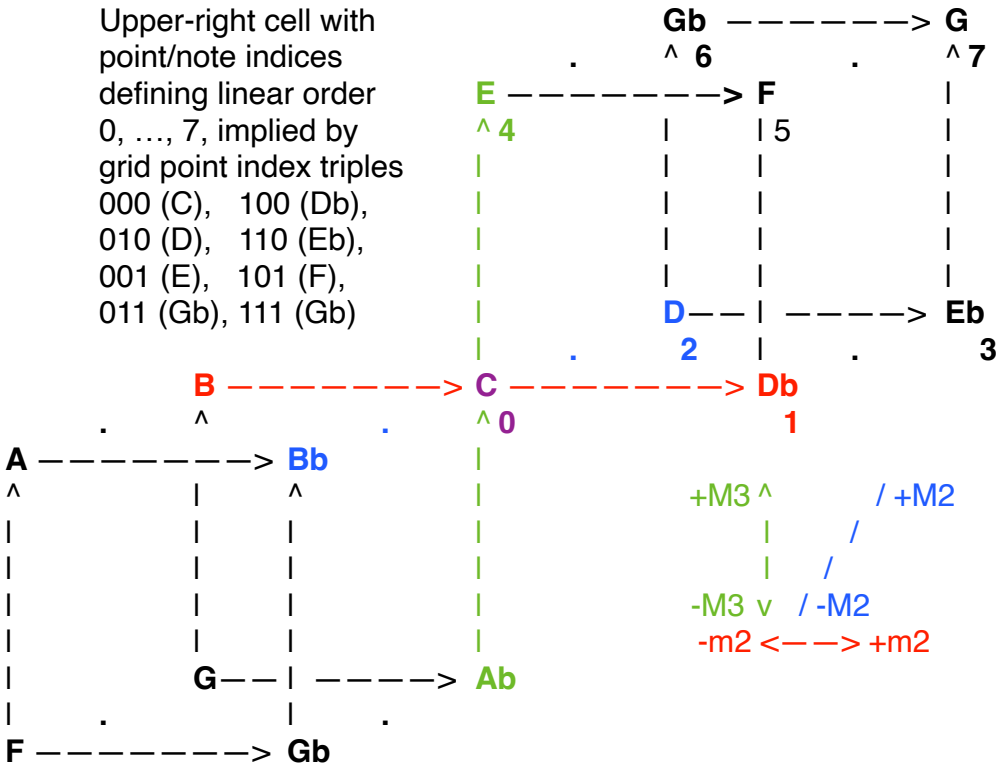
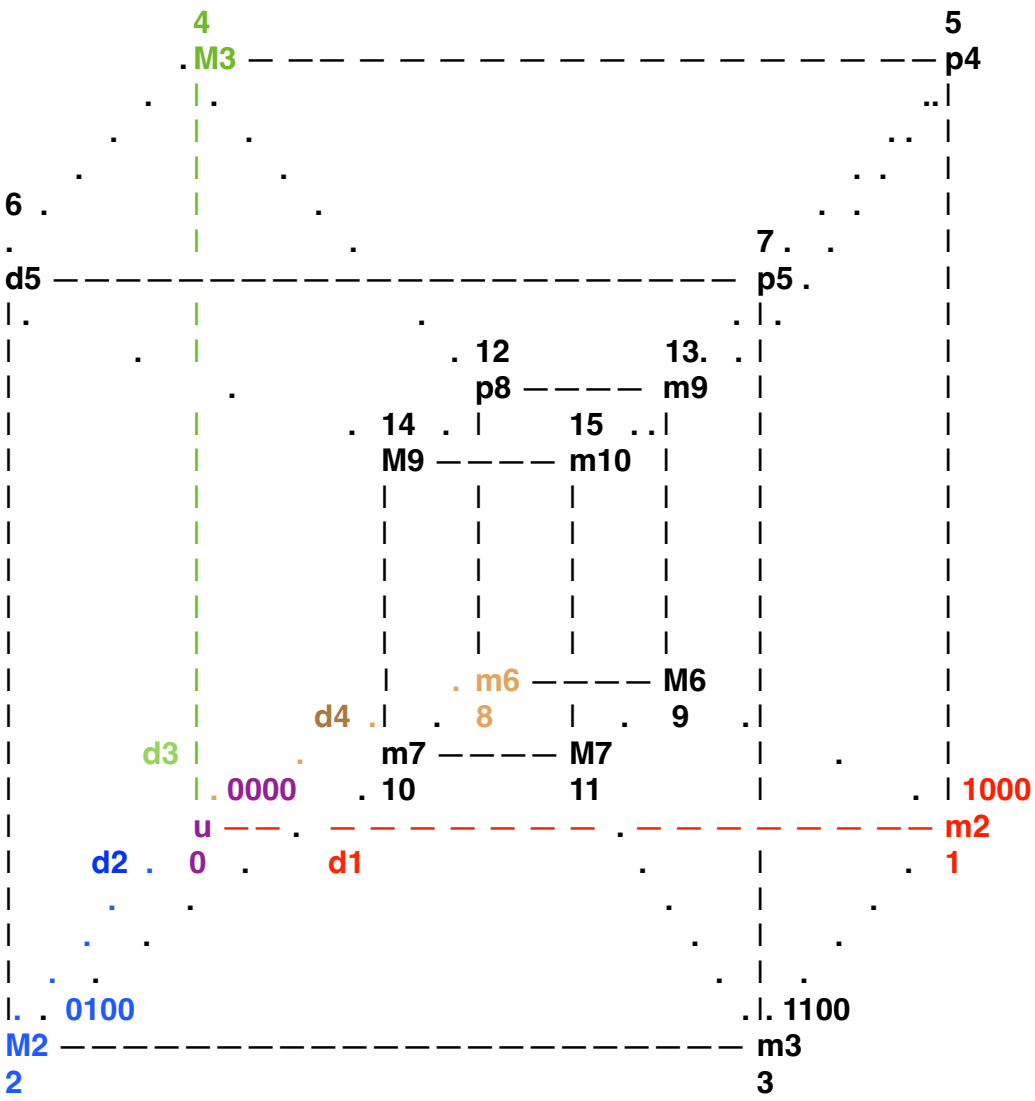


Figure showing note intervals as grid point annotations (unison = 'u') with the resulting linear-order decimal indices, from 0 to 15. Indices are written next to intervals at each grid cell corner point. The decimal integer values

are obtained by converting the 2^4=16 quadruple binary point multi-indices which, in this situation, must be interpreted as “reverse binary notation,” i.e., where the least (most) significant bit is the leftmost (rightmost) bit, implying

0000	= 0,	1000	= 1,	0100	= 2,	1100	= 3,
0010	= 4,	1010	= 5,	0110	= 6,	1110	= 7,
0001	= 8,	1001	= 9,	0101	= 10,	1101	= 11,
0011	= 12,	1011	= 13,	0111	= 14,	1111	= 15.



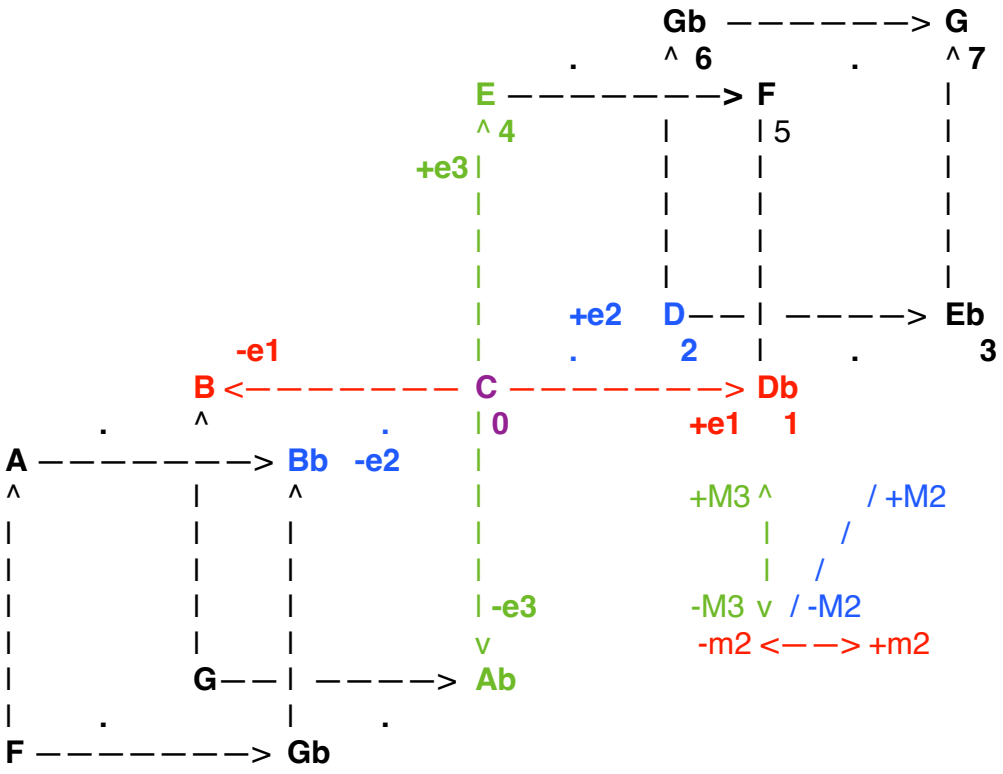
The goal is to determine four specific note intervals for the local directions (1,0,0,0), (0,1,0,0), (0,0,1,0) and (0,0,0,1) such that the possible 2^4 = 16 forward movements from the base note with index quadruple 0000, to the 16 available points include all intervals within an octave, i.e., unison (u), m2, M2, m3, M3, p4, p5, m6, M6, m7, M7 and p8. To achieve this goal, we can associate m2, M2, M3 and m7 note intervals with movement directions

(1,0,0,0), (0,1,0,0), (0,0,1,0) and (0,0,0,1) as shown in the figure. We can subsequently annotate the grid points with the notes obtained when moving to them via one of 16 possible note intervals from the base note C.

The tonnetz uses 2x2 grid cells for its underlying rectilinear grid, permitting only the representation and visualization of four note intervals, i.e., u, m3, M3 and p5. **The 16-point grid cell in the 4-dimensional case shown in the figure makes it possible to represent and visualize the 16 note intervals u, m2, M2, m3, …, p8, m9, M9 and m10 in a single grid cell.** The grid cell shown here is the "prototype cell," since the specific note associated with grid point (0,0,0,0), having base note index 0, can be any note of the chromatic scale. Thus, all forward movements, all "jumps," from any base note by the intervals u, m2, M2, m3, …, p8, m9, M9 or m10, are represented by a single cell.

A 4-dimensional single hyper-cube lattice suffices to capture all possible note intervals that can be defined over the base note C, for example. Many combinatorially possible k-note chords can be constructed and represented via sequences of movements ("forward jumps") in positive direction (+) of one of the four grid directions or one of the diagonals (= sums of forward movements using grid directions) of 2-dimensional and "3-dimensional faces" of the 4-dimensional hyper-cube defined by the local 16-point, single hyper-cube stencil.

In the 3-dimensional case, one can determine 1-, 2- and 3-note chords in the local rectilinear 8-point single-cell grid stencil. All combinatorially possible chords are obtained by defining paths, e.g., obtained by moving forward (or possibly also backward) in any of the directions available at grid points/notes. In the presented case one, one can move forward by m2, M2 and M3 intervals in the three positive grid directions, In addition, one can consider moving in a more complex fashion by using the forward/backward (+/-) directions given by diagonals of faces of cells and diagonals in the interior of cells etc. When considering forward (+) and backward (-) movements and the combinatorially possible directions/direction vectors for individual movements, one can derive the complete list of movements and resulting chords, see the figure and associated table below. The letters 'n', 'e', 'f' and 'c' are used to abbreviate the terms "null", "edge", "face" and "cell," respectively.



Moving to	Movement type/name	Added movements written as sums	Interval and resulting chord	3-note chord via stacking
Null movement (1)				
(0,0,0)	n	n	u : C-C	
Edge movements (6)				
(1,0,0)	e1	e1	m2 : C-Db	
(-1,0,0)	-e1	-e1	-m2 : Bb-C	Bb-C-Db
(0,1,0)	e2	e2	M2 : C-D	
(0,-1,0)	-e2	-e2	-M2 : B-C	B-C-D
(0,0,1)	e3	e3	M3 : C-E	
(0,0,-1)	-e3	-e3	-M3 : Ab-C	Ab-C-E

Face movements (12)				
(1,1,0)	f12	e1 + e2	m3 : C-Eb	
(-1,-1,0)	f-1-2	-e1 + -e2	-m3 : A-C	A-C-Eb
(1,0,1)	f13	e1 + e3	p4 : C-F	
(-1,0,-1)	f-1-3	-e1 + -e3	-p4 : G-C	G-C-F
(0,1,1)	f23	e2 + e3	d5 : C-Gb	
(0,-1,-1)	f-2-3	-e2 + -e3	-d5 : Gb-C	Gb-C-Gb
(-1,1,0)	f-12	-e1 + e2	m2 : C-Db	
(1,-1,0)	f1-2	e1 + -e2	-m2 : B-C	B-C-Db
(-1,0,1)	f-13	-e1 + e3	m3 : C-Eb	
(1,0,-1)	f1-3	e1 + -e3	-m3 : A-C	A-C-Eb
(0,-1,1)	f-23	-e2 + e3	M2 : C-D	
(0,1,-1)	f2-3	e2 + -e3	-M2 : Bb-C	Bb-C-D
Cell (cell-interior) movements (8)				
(1,1,1)	c123	e1 + e2 + e3	p5 : C-G	
(-1,-1,-1)	c-1-2-3	-e1 + -e2 + -e3	-p5 : F-C	F-C-G
(-1,1,1)	c-123	-e1 + e2 + e3	p4 : C-F	
(1,-1,-1)	c1-2-3	e1 + -e2 + -e3	-p4 : G-C	G-C-F
(1,-1,1)	c1-23	e1 + -e2 + e3	m3 : C-Eb	
(-1,1,-1)	c-12-3	-e1 + e2 + -e3	-m3 : A-C	A-C-Eb
(1,1,-1)	c12-3	e1 + e2 + -e3	-m2 : B-C	
(-1,-1,1)	c-1-23	-e1 + -e2 + e3	m2 : C-Db	B-C-Db

Forward (+) and backward (-) movements along edges, face diagonals and cell (cell-interior, cube-interior) diagonals, starting on base note C, producing the listed 1 + 6 + 12 + 8 = 27 = 3^3 possibilities. The column “3-note chord via stacking” shows the results of stacking, or “compositing,” the last two 2-note chords to define a 3-note chord, listing notes of the 3-note chords in linear order, from lowest to highest note.

While the table only lists these specific 3-note chords, one can also produce the chords captured in the standard tonnetz. The table includes the note intervals m3, -m3, M3, -M3 of the tonnetz, allowing one to generate all chords obtainable via stacking these minor and major thirds. Further, by extending such a local rectilinear grid stencil in the local neighborhood, around a local origin with an associated base

note, and by using rectilinear grids of higher dimensions, one can represent more chords — some of them being harmonically "meaningful," some of them not being harmonically "meaningful." **The ability to represent, analyze and construct a multitude of chords and chords of increasing complexity is one of the major objectives for generalizing the standard tonnetz of Leonhard Euler to more general higher-dimensional *tone lattices*.**

Considering the 3-dimensional case once more, every movement can be written as a sum of movements along grid axes, i.e., along the three axes with the six direction vectors e1, e2, e3 and -e1, -e2, -e3. Counting the null-movement, the total number of movement directions is 1+2x3+2x6+2x4 = 1+6+12+8 = 27 = 3^3, which is the number of grid points of the 3x3x3 local grid stencil. This total number results, since 27 directions/direction vectors are defined by the "view vectors" defined when "viewing" from C at any one of the 27 stencil points in the local neighborhood, including the view from C to C itself (null vector). Movements along the three grid axes have the corresponding six intervals m3(e1), M3(e2) and m2(e3) for the three forward movements and -m3(-e1), -M3(-e2) and -m2(-e3) for the three backward movements. Further, one can create names for diagonal movements in cell faces and diagonal movements in the interior of a cell (cube). This has been done for the table above. Four simple examples are the following four stack results that only stack m3 and M3 note intervals within the bottom layer/face of a cube, using C as base note:

m3m3 —> C-Eb-Gb = Cdim, m3M3 —> C-Eb-G = Cmin,
M3m3 —> C-E-G = Cmaj, M3M3 —> C-E-Gb = Caug.

By considering larger grid stencils in the neighborhood of a selected base note placed at the origin of a rectilinear grid one can identify chords consisting of any number of notes, in principle. These chords are obtained, in the case of an extended 3-dimensional rectilinear grid with intervals m3, M3 and m2 associated with its three axes (in positive direction), by stacking via m3, M3 and m2 interval sequences. Intervals are associated with the axis directions of the rectilinear grid, with face diagonals of cells and with cell (cell-interior) diagonals. Thus, k-note chords can be obtained by stacking in an extended grid neighborhood, i.e., an extended *tone lattice*, around a base note. These k-note chords are polylines consisting of line segments being grid edges, face diagonals or cell-interior diagonals. Geometrically, these polylines are "zigzag" lines, connecting points of the rectilinear grid in a rather general way.

We return to the 4-dimensional case again. Here, considering only a single local grid cell with local point indexing from 0000 (0) to 1111 (16), one can move from a base note C at grid point 0000 via 16 forward (+) movements to 16 notes, by moving forward from C (0000 = 0) in one movement — keeping in mind that all such movements can be decomposed into sequences of only forward (+) movements, referred to as e1, e2, e3 and e4 movement types, i.e., all movements in the rectilinear grid cell that follow the four forward (+) grid directions. The following table summarizes the 16 possibilities.

Moving to	Movement type/name	Added movements written as sums	3-note chord via stacking
Null movement (1)			
(0,0,0,0)	n	n	u : C-C
Edge movements (4)			
(1,0,0,0)	e1	e1	m2 : C-Db
(0,1,0,0)	e2	e2	M2 : C-D
(0,0,1,0)	e3	e3	M3 : C-E
(0,0,0,1)	e4	e4	m6 : C-Ab
Face movements (6)			
(1,1,0,0)	f12	e1 + e2	m3 : C-Eb
(1,0,1,0)	f13	e1 + e3	p4 ; C-F
1,0,0,1)	f14	e1 +e4	M6 ; C-A
(0,1,1,0)	f23	e2 + e3	d5 ; C-Gb
(0,1,0,1)	f24	e2 + e4	m7 : C-Bb
(0,0,1,1)	f34	e3 + e4	p8 : C-C’
Cell-interior movements (4)			
(1,1,1,0)	c123	e1 + e2 + e3	p5 : C-G
(1,1,0,1)	c124	e1 + e2 + e4	M7 : C-B
(1,0,1,1)	c134	e1 + e3 +e4	m9 : C-Db’
(0,1,1,1)	c234	e2 + e3 + e4	M9 : C-D’
Hyper-cell-interior movement (1)			
(1,1,1,1)	h1234	e1 + e2 + e3 + e4	m10 : C-Eb’

The 16-point prototype cell (cube) with base note C defines 16=2^4 forward (+) paths: 1 null movement, 4 edge movements, 6 face(-diagonal) movements, and 4 cell (cell-interior) diagonal movements and 1 hyper-cell (hyper-cell-interior) diagonal) movement. These numbers arise as a 4-dimensional hyper-cube has 1 0-dimensional cell (origin), 4 1-dimensional cells (edges), 6 2-dimensional cells (faces), 4 3-dimensional cells (“standard cubes”) and 1 4-dimensional “hyper-cell.” When considering an extended point stencil around a local origin of a grid point with index quadruple (0,0,0,0) and deriving all possible local index quadruples ranging from (-1,-1,-1,-1) to (1,1,1,1) in the origin’s (base note’s) neighborhood, one obtains a neighborhood of 3^4 = 81 points — as a consequence of using three index-component values -1, 0 and 1 and quadruple multi-indices. It is beyond the scope and purpose of these notes to explore such a truly complex neighborhood or to determine details about chords and chord relationships that can be represented with it.

Conclusions. The same ideas can be used for the construction of 5-, 6- or 7-dimensional *tone lattices*, where each single cell of an rectilinear grid is a 5-, 6- or 7-dimensional hyper-cube with 2^5 = 32, 2^6 = 64 or 2^7 = 128 corner points. While such high-dimensional structures and lattices would be hard to visualize in an easily comprehensible way, they might nevertheless be useful for computational approaches for the representation, analysis and design of complex musical harmony. Thus, with increasing number of dimensions it might be proper to use automated computational, algorithmic methods for *tone lattices*.

The choice of a tone lattice of a specific dimension and size/resolution and the set of notes used for annotation depends on the specific musical "category and genre" — e.g., ancient Greek music (tetrachords), medieval church music (church modes, Hildegard von Bingen, Gregorian chants), Western classical music (modern minor/major scales, diatonic/chromatic scales), jazz (Blues scales, pentatonic sales) — and the goals one has in mind for the analysis of complex harmony or when exploring harmony for composition. Formally, there exist 12C0 + 12C1 + 12C2 + … + 12C12 = 2^12 = 4096 chords for the 12-note C, Db, D, …, Bb, B one can study.

These notes sketch approaches for generalizing the 2-dimensional **tonnetz** of **Leonhard Euler** to higher-dimensional representations via lattices defined by high-dimensional rectilinear grids as underlying geometrical structure. Thus, the notes refer to these rectilinear grids and structures as ***tone lattices***.

Of course, the notes contain errors that will hopefully be forgiven by a critical reader. A person with training in music or mathematics will see that the notation used is not following general, rigorous principles and style of academic music theory or mathematics. Nevertheless, the general ideas described should be understandable. The notes discuss key concepts several times, usually using slightly different examples or language. In other words, the same fact, idea or principle is often presented in a variety of ways — e.g., by using synonymous terms for the same idea or object — borrowed from scientific, "practical" or colloquial language. This is done in order to reach a diverse readership via a multitude of styles of expression. The chosen exposition could be viewed as "redundant presentation and writing," but the author simply intends to express and emphasize crucial ideas via a "layman style," a *sui generis* style that can hopefully reach those with a sincere interest in the topics covered in the notes but not having a "deep" professional training in music theory or mathematics. The author hopes that the notes stimulates readers to explore interesting aspects further, in much more detail. Readers will quickly determine that two skills are required: adding 1 to a number and subtracting 1 from a number.

May these incomplete notes inspire additional studies of many meaningful, exciting and often surprising relationships between music and mathematics.

References

[1] Euler, Leonhard, 1739, Tentamen novae theoriae musicae ex certissismis harmoniae principiis dilucide expositae (An attempt at a new theory of music, exposed in all clearness, according to the most well-founded principles of harmony), St. Petersburg Academy of Sciences, Russia, p. 147.

[2] Banchoff, Thomas F., 1990, Beyond the Third Dimension — Geometry, Computer Graphics and Higher Dimensions, Scientific American Library, W. H. Freeman and Company, New York, New York.