Chapter 4

Curvature approximation for triangulated surfaces and trivariate functions

4.1. Introduction and essential terms of differential geometry

Methods for exactly calculating and approximating curvatures are important in geometric modeling for two reasons. In order to judge the quality of a surface one commonly computes curvatures for points on the surface, renders the surface's curvature as a texture map onto the surface and can thereby detect regions with undesired curvature behavior, such as surface regions locally changing from an elliptic to a hyperbolic shape. On the other hand, surface schemes are being developed requiring higher order geometric information as input, e.g., normal vectors and normal curvatures.

Definitions and theorems from classical differential geometry are reviewed as far as they are needed for the proceeding. In classical differential geometry a surface is understood as a mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$,

$$\mathbf{x}\,(\mathbf{u}) \;=\; \big(\, x(u,v), y(u,v), z(u,v)\,\big)^T \;\subset\; \mathbb{R}^3, \quad \mathbf{u} \in D \;\subset\; \mathbb{R}^2. \tag{4.1.}$$

The standard formulae are then used to derive techniques for approximating normal curvatures when a two-dimensional triangulation of a finite point set with associated outward unit normal vectors is given in three-dimensional space. Consequently, curvature estimates can be incorporated into existing surface generating schemes allowing curvature input. The quality of the curvature approximation is tested for triangulated surfaces obtained from a known parametric surface $\mathbf{x}(\mathbf{u})$.

The theory of two-dimensional surfaces can easily be extended to the case of three-dimensional surfaces, e.g., graphs of trivariate functions approximating scalar fields over a three-dimensional domain,

$$\left(\mathbf{x}^T, f(\mathbf{x}) \right)^T = \left(x, y, z, f(x,y,z) \right)^T \subset I\!R^4, \quad \mathbf{x} \in D \subset I\!R^3. \qquad (4.2.)$$

If the approximating function $f(\mathbf{x})$ is known, normal curvatures for its graph can be computed accurately, thus allowing to visualize the graph's curvature behavior using one of the rendering techniques for trivariate data sets introduced in chapter 2. Qualitative changes in f's three-dimensional graph in four-dimensional space can be observed, hence providing a quality measure for the chosen approximation method.

Future trivariate scattered data approximation schemes might as well require input such as normal curvatures when the approximation process is seen from a more geometric point of view interpreting the result as a three-dimensional hypersurface. An estimation method is presented for approximating normal curvatures at four-dimensional points $\left(x_i, y_i, z_i, f(x_i, y_i, z_i) \right)^T$ on a three-dimensional hypersurface in order to generate a smooth graph obtained by solving the trivariate approximation problem. Again, the quality of the curvature estimation technique is tested for known trivariate functions. Possibly, multivariate approximation schemes for even higher dimensions ($f(x_1, ..., x_n)$, $n \geq 4$) will consider such geometric information shortly.

Good introductions to differential geometry are [Brauner '81], [do Carmo '76], [Lipschutz '80], [Strubecker '55,'58,'59], and [Struik '61]. Differential geometry is treated more analytically in [O'Neill '69]. One of the most comprehensive works on this subject is [Spivak '70]. Some information can also be found in [Farin '88]. An example for estimating curvatures from a discrete point set is [Calladine '86]. There, a technique for approximating Gaussian curvature for points in a two-dimensional triangulation in three-dimensional space is discussed. An example for a surface scheme allowing curvature input is introduced in [Hagen & Pottmann '89]; a triangular surface scheme is described considering positional, normal vector, and normal curvature information.

Definition 4.1. A **regular parametric two-dimensional surface** of class C^m ($m \geq 1$) is the point set S in real three-dimensional space $\mathbb{R}^3$ defined by the mapping

$$\mathbf{x} = \mathbf{x}(\mathbf{u}) = \big(x(u,v), y(u,v), z(u,v) \big)^T \tag{4.3.}$$

of an open set $U \subset \mathbb{R}^2$ into $\mathbb{R}^3$ such that

(i) all partial derivatives of x, y, and z of order m or less are continuous in U, and

(ii) $\mathbf{x}_u \times \mathbf{x}_v \neq (0,0,0)^T$ for all $(u,v) \in U$

(the subscripts u and v indicating partial differentiation with respect to u and v, respectively).

Since condition (ii) in Definition 4.1. implies the linear independence of the two vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ at any point on the regular surface, they determine the tangent plane at every surface point.

Definition 4.2. The **tangent plane** at a point $\mathbf{x}_0 = \mathbf{x}(\mathbf{u}_0)$ on a regular parametric two-dimensional surface in three-dimensional space is defined as the set of all points $\mathbf{y}$ in $I\!R^3$ satisfying the equation

$$\mathbf{y} \;=\; \mathbf{x}_0 + a\mathbf{x}_u(\mathbf{u}_0) + b\mathbf{x}_v(\mathbf{u}_0), \qquad a, b \in I\!R. \tag{4.4.}$$

Definition 4.3. The **outward unit normal vector** $\mathbf{n}_0 = \mathbf{n}(\mathbf{u}_0)$ of a regular parametric surface at a point $\mathbf{x}_0$ is given by

$$\mathbf{n}_0 \;=\; \frac{\mathbf{x}_u(\mathbf{u}_0) \times \mathbf{x}_v(\mathbf{u}_0)}{||\mathbf{x}_u(\mathbf{u}_0) \times \mathbf{x}_v(\mathbf{u}_0)||} \;=\; \frac{\mathbf{x}_u \times \mathbf{x}_v}{||\mathbf{x}_u \times \mathbf{x}_v||}, \tag{4.5.}$$

where $||\cdot||$ indicates the Euclidean norm.

Definition 4.4. Let $\mathbf{x}(\mathbf{u})$ be a regular parametric surface of class m, $m \geq 2$, and $\mathbf{c}(t) = \mathbf{c}\big(u(t), v(t)\big)$ be a (regular) curve of class 2 on the surface through the point $\mathbf{x}_0 = \mathbf{x}(\mathbf{u}_0)$. The **normal curvature vector** to $\mathbf{c}(t)$ at $\mathbf{x}_0$ is the projection of the curvature vector $\mathbf{k} = \dot{\mathbf{t}}/||\dot{\mathbf{t}}||$, $\mathbf{t} = \dot{\mathbf{c}}/||\dot{\mathbf{c}}||$, onto the unit surface normal vector $\mathbf{n}_0$,

$$\mathbf{k}_n \;=\; (\mathbf{k} \cdot \mathbf{n}_0)\; \mathbf{n}_0. \tag{4.6.}$$

The proportionality factor $\mathbf{k} \cdot \mathbf{n}_0$ is called the **normal curvature**, denoted by κ_n.

Definition 4.5. The second degree polynomial

$$I\;(du, dv) \;=\; \mathbf{x}_u \cdot \mathbf{x}_u\; du^2 + 2\; \mathbf{x}_u \cdot \mathbf{x}_v\; du\; dv + \mathbf{x}_v \cdot \mathbf{x}_v\; dv^2$$

$$= E\ du^2 + 2F\ du\ dv + G\ dv^2, \qquad (4.7.)$$

where $du, dv \in I\!R$, is called the **first fundamental form** of a regular parametric surface $\mathbf{x}(\mathbf{u})$. The coefficients E, F, and G are called the **first fundamental coefficients**.

Definition 4.6. Assuming that the regular parametric surface $\mathbf{x}(\mathbf{u})$ is at least of order 2, the second degree polynomial

$$II\ (du, dv) \ = \ -\mathbf{x}_u \cdot \mathbf{n}_u\ du^2 - (\mathbf{x}_u \cdot \mathbf{n}_v + \mathbf{x}_v \cdot \mathbf{n}_u)\ du\ dv - \mathbf{x}_v \cdot \mathbf{n}_v\ dv^2$$

$$= \mathbf{x}_{uu} \cdot \mathbf{n}\ du^2 + 2\ \mathbf{x}_{uv} \cdot \mathbf{n}\ du\ dv + \mathbf{x}_{vv} \cdot \mathbf{n}\ dv^2 \ = \ L\ du^2 + 2M\ du\ dv + N\ dv^2, \quad (4.8.)$$

where $du, dv \in I\!R$, is called the **second fundamental form** of $\mathbf{x}(\mathbf{u})$. The coefficients L, M, and N are called the **second fundamental coefficients**.

Definition 4.7. The two (real) eigenvalues κ_1 and κ_2 of the matrix

$$-A \ = \ -\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \ = \ \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}, \qquad (4.9.)$$

where

$$a_{1,1} \ = \ \frac{MF - LG}{EG - F^2}, \qquad a_{1,2} \ = \ \frac{LF - ME}{EG - F^2},$$
$$a_{2,1} \ = \ \frac{NF - MG}{EG - F^2}, \qquad a_{2,2} \ = \ \frac{MF - NE}{EG - F^2},$$

of a regular surface of class of at least 2 at a point $\mathbf{x}_0$ are called **principal curvatures** of the regular parametric surface at $\mathbf{x}_0$. The associated eigenvectors determine the **principal curvature directions**. Therefore, the principal curvatures are the (real) roots of the characteristic polynomial of $-A$, the quadratic polynomial

$$\kappa^2 + (a_{1,1} + a_{2,2})\ \kappa + a_{1,1}a_{2,2} - a_{1,2}a_{2,1}. \qquad (4.10.)$$

Remark 4.1. The equations for the matrix elements $a_{i,j}$ in equation (4.9.) are known as the Gauss-Weingarten equations (or the Gauss-Weingarten map).

Remark 4.2. It is shown in [Spivak '70] that the eigenvalues of the matrix $-A$ in equation (4.9.) are always real, and the associated eigenvectors are orthogonal to each other.

Definition 4.8. The average H of the two principal curvatures κ_1 and κ_2 is called the **mean curvature**, the product K is called the **Gaussian curvature** of the regular parametric surface $\mathbf{x}(\mathbf{u})$ at $\mathbf{x}_0$,

$$H = \frac{1}{2}(\kappa_1 + \kappa_2), \qquad K = \kappa_1 \kappa_2. \tag{4.11.}$$

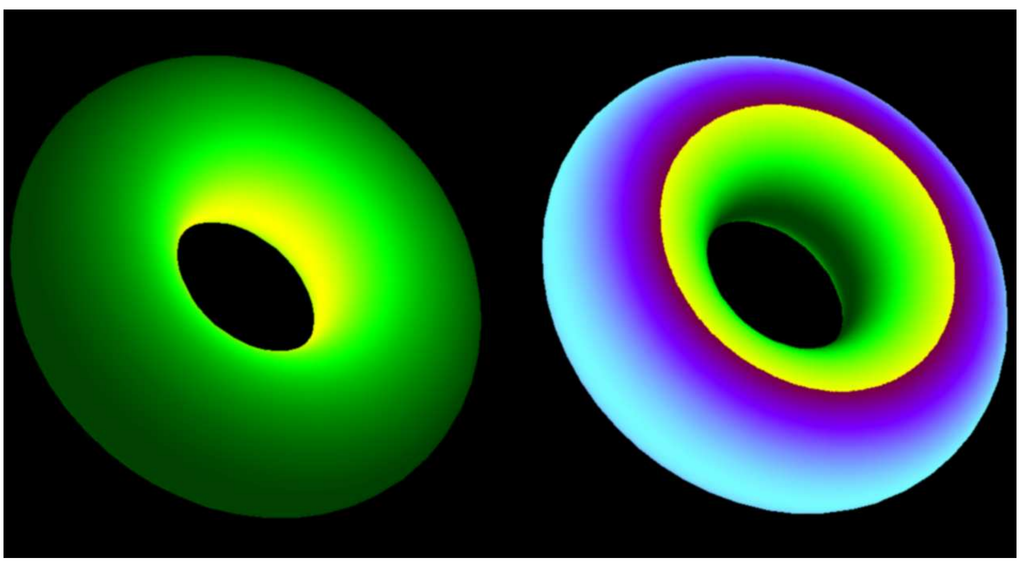

Fig.4.1. Texture map of mean and Gaussian curvature onto a torus,
$\left((2+\cos u)\cos v, (2+\cos u)\sin v, \sin u \right)^T$, $u, v \in [0, 2\pi]$;
green/yellow representing negative values,
magenta/blue representing positive values.