Chapter 5

Data reduction for triangulated surfaces

5.1. Existing schemes and necessary definitions

Data reduction schemes are essential for efficient data storage. Storing and processing more data than necessary is both a waste of space and time. In the context of digitizing curves and surfaces an efficient scheme does not generate more data points than necessary to represent a particular geometric object within a prescribed tolerance. E.g., when using piecewise linear approximation storing lots of data points in "flat" regions is rather unsophisticated.

Based on this observation a data reduction algorithm is developed. Given a two-dimensional triangulation in three-dimensional space each triangle is weighted according to the principal curvatures at its vertices. A triangle indicates a surface region with low curvature, when the sum of the absolute curvatures at its vertices is low. This measure is used as a weight to determine a triangle's significance in the triangulation.

The lower a triangle's weight is the earlier it is removed. This paradigm is applied to derive an iterative algorithm removing the triangle with the lowest weight (the lowest absolute curvatures) in each step. Thus, the triangulation is adaptively modified, and the local density of triangles reflects the original surface's curvature behavior. At the end, surface regions with low curvature are represented by relatively larger triangles than highly curved regions.

The term "data-dependent triangulation" is commonly used when function values at data points in the plane are considered for constructing a "good" triangulation of the implied piecewise linear function. This concept is discussed in [Dyn et al. '90a] and [Dyn et al. '90b]. Knot removal strategies for spline curves and tensor product surfaces (in the function setting) are described in [Arge et al. '90], [Lyche & Mørken '87], and [Lyche & Mørken '88]. Given scattered points in the plane and associated function values an iterative knot removal algorithm is discussed in [Le Méhauté & Lafranche '89] based on the resulting spline.

The data reduction technique introduced here is similar to the method in [Le Méhauté & Lafranche '89] in the sense that an iterative reduction scheme is used. However, the method deduced subsequently removes triangles instead of single data points. Furthermore, it is not restricted to a two-dimensional triangulation obtained as the graph of a bivariate function, but can be applied to more general two-dimensional triangulations in three-dimensional space, e.g., triangulations of parametric surfaces and of contours of trivariate functions.

The triangle removal algorithm allows the user to specify a percentage of the original number of triangles determining the number of triangles to be removed. Alternatively, it is possible to have the reduction process terminate automatically when a certain error tolerance is exceeded. This, however, can only be done for a triangulation obtained from a bivariate function's graph. Such a triangulation allows to compute the error introduced during each reduction step, since both the initial triangulation and the triangulation at a certain iteration step are piecewise linear functions, and their difference can easily be measured in ordinate-direction.

In principle, the method can be extended to the reduction of three-dimensional surface triangulations obtained as triangulations of three-dimensional graphs of trivariate functions in four-dimensional space. A principal curvature approximation scheme for such hypersurfaces has already been introduced in chapter 4.3. However, this extension is not investigated.

Before describing the iterative triangle removal algorithm, some necessary definitions are introduced.

Definition 5.1. Given a two-dimensional triangulation in two- or three-dimensional space, the **triangle platelet** $\mathcal{TP}_i$ associated with a triangle T_i (identified with the index triple (v_1^i, v_2^i, v_3^i) specifying its vertices) in the triangulation is the set of all triangles T_j (identified with their index triples (v_1^j, v_2^j, v_3^j)) sharing at least one of T_i's vertices as a common vertex,

$$\mathcal{TP}_i = \bigcup \left\{ T_j = (v_1^j, v_2^j, v_3^j) \mid v_k^i = v_1^j \vee v_k^i = v_2^j \vee v_k^i = v_3^j,\ k = 1,2,3 \right\}. \quad (5.1.)$$

The triangle platelet $\mathcal{TP}_i$ is the set of triangles in a two-dimensional triangulation affected by the removal of the triangle T_i.

Definition 5.2. The set of triangles

$$\mathcal{CP}_i \;=\; \mathcal{TP}_i \setminus \{T_i\} \tag{5.2.}$$

is called the **corona** of the triangle platelet $\mathcal{TP}_i$.

Definition 5.3. The **corona** $\mathcal{CP}_i$ is **continuous** if for each pair of triangles T_{l_1}, $T_{l_m} \in \mathcal{CP}_i$ triangles $T_{l_2}, ..., T_{l_{m-1}} \in \mathcal{CP}_i$ exist such that

$$\bigwedge_{i=1}^{m-1} \left(T_{l_i} \text{ and } T_{l_{i+1}} \text{ are neighbors } \right); \tag{5.3.}$$

otherwise, the **corona** is **discontinuous**.

Definition 5.4. The **corona** $\mathcal{CP}_i$ is **cyclic** if it contains triangles T_{l_0}, T_{l_1}, and T_{l_2} such that

$$\bigwedge_{i=0}^{2} \left(T_{l_i} \text{ and } T_{l_{((i+1) \bmod 3)}} \text{ are neighbors } \right); \tag{5.4.}$$

otherwise, the **corona** is **acyclic**.

Figure 5.1. illustrates a triangle platelet $\mathcal{TP}_i$ with a continuous and an discontinuous corona and a cyclic corona.

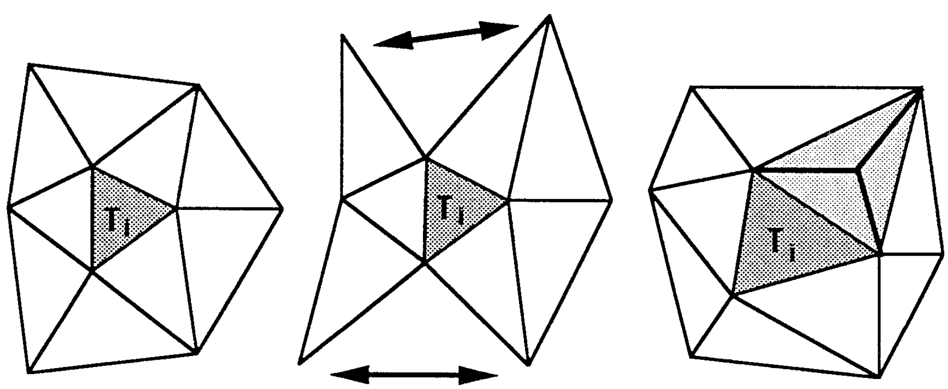

Fig. 5.1. Triangle platelet with continuous and discontinuous corona and cyclic corona.

Definition 5.5. The **corona** $\mathcal{CP}_i$ is **closed** if each triangle in $\mathcal{CP}_i$ has exactly two neighbors also elements of $\mathcal{CP}_i$; otherwise, the **corona** is called **open**.

Theorem 5.1. *Denoting the elements of a continuous, acyclic corona $\mathcal{CP}_i$ by $T_{l_0}, ..., T_{l_{m_i-1}}$, an order can be imposed on this set of triangles. If the corona $\mathcal{CP}_i$ is closed any triangle $T_{l_j} \in \mathcal{CP}_i$ among T_i's neighbors can be chosen as the first triangle T^0 of the ordered set $\mathcal{CP}_i^{ord}$. If $\mathcal{CP}_i$ is open a triangle $T_{l_j} \in \mathcal{CP}_i$ not having more than one neighbor in $\mathcal{CP}_i$ is selected as the first triangle T^0 of $\mathcal{CP}_i^{ord}$. The set $\mathcal{CP}_i^{ord}$ is generated by computing the sequence of sets*

$$\mathcal{S}_0 \;=\; \{T^0\},$$

$$\mathcal{S}_k \;=\; \mathcal{S}_{k-1} \,\cup\, \{T^k\} \;=\; \{T^0, ..., T^{k-1} \mid T^j \textit{ precedes } T^{j+1},\; j = 0...k-2\ \} \,\cup\, \{T^k\},$$

$$k = 1...m_i - 1, \tag{5.5.}$$

where $T^k \in \mathcal{CP}_i$, $T^k \notin \mathcal{S}_{k-1}$, and T^k is a neighbor of T^{k-1}. The final ordered set $\mathcal{CP}_i^{ord}$ equals $\mathcal{S}_{m_i-1}$.

Proof. Trivial.

Remark 5.1. If the set of triangles to be ordered is a closed corona then the last triangle T^{m_i-1} precedes the first triangle T^0 as well.

Definition 5.6. Denoting the set of vertices in $\mathcal{TP}_i$ by $\{\mathbf{x}_{l_0}, ..., \mathbf{x}_{l_{N_i}}\}$ such that $\mathbf{a}_0 = \mathbf{x}_{l_0}$, $\mathbf{a}_1 = \mathbf{x}_{l_1}$, and $\mathbf{a}_2 = \mathbf{x}_{l_2}$ are T_i's vertices (in counterclockwise order), the set

$$\mathcal{B}_i \;=\; \{\ \mathbf{x}_{l_j} \mid j = 3...N_i\ \} \tag{5.6.}$$

is called the **boundary vertex set** of the triangle platelet $\mathcal{TP}_i$.

Theorem 5.2. *Given the ordered set of triangles $\mathcal{CP}_i^{ord}$ of a continuous, acyclic corona an order for the boundary vertex set $\mathcal{B}_i$ is implied. If the first triangle $T^0 \in \mathcal{CP}_i^{ord}$ is a neighbor of T_i the vertex of T^0 not being a vertex of T_i is chosen as the first boundary vertex $\mathbf{y}_0$ of the ordered set $\mathcal{B}_i^{ord}$. If the first triangle $T^0 \in \mathcal{CP}_i^{ord}$ is not a neighbor of T_i the vertex of T^0 neither being a vertex of T_i nor of T^1 is chosen as the first boundary vertex $\mathbf{y}_0$. The set $\mathcal{B}_i^{ord}$ is generated by computing the sequence of sets*

$$\begin{aligned} \mathcal{S}_0 &= \{\mathbf{y}_0\}, \\ \mathcal{S}_k &= \mathcal{S}_{k-1} \cup \{\mathbf{y}_j\}, \qquad k = 1...m_i - 1, \end{aligned} \tag{5.7.}$$

where $\mathbf{y}_j$ is a vertex of T^k, $\mathbf{y}_j$ is not a vertex of T_i, and $\mathbf{y}_j \notin \mathcal{S}_{k-1}$. The final ordered set $\mathcal{B}_i^{ord} = \{\mathbf{y}_0, ..., \mathbf{y}_{n_i}\}$ equals S_{m_i-1}.

Proof. Trivial.

Definition 5.7. The polygon formed by the directed line segments

$$\overline{\mathbf{y}_j \mathbf{y}_{(j+1) \bmod (n_i+1)}}, \qquad j = 0...N, \tag{5.8.}$$

where N equals $n_i - 1$ (open corona) or n_i (closed corona), is called the **platelet boundary polygon** of the triangle platelet $\mathcal{TP}_i$.

In order to ensure that the orientation of the platelet boundary polygon has the same orientation as the triangle T_i given by the line segments $\overline{\mathbf{a}_j \mathbf{a}_{(j+1) \bmod 3}}$, $j = 0, 1, 2$, the next definition is needed.

Definition 5.8. The (ordered) **boundary vertex set** $\mathcal{B}_i^{ord}$ is **ordered counterclockwise** if the platelet boundary polygon satisfies the following condition: If $\mathbf{y}_j$ and $\mathbf{y}_{(j+1) \bmod (n_i+1)}$, $j = 0...N$, are the end points of a line segment of the platelet boundary polygon and there are edges in $\mathcal{TP}_i$ connecting $\mathbf{y}_j$ with the vertex $\mathbf{a}_k$, $k \in \{0, 1, 2\}$, and $\mathbf{y}_{(j+1) \bmod (n_i+1)}$ with a different vertex $\mathbf{a}_l$, $l \in \{0, 1, 2\}$, then it is $k = 0$ and $l = 1$, or $k = 1$ and $l = 2$, or $k = 2$ and $l = 0$.

If the condition stated in Definition 5.8. is violated by the order imposed on a boundary vertex set $\mathcal{B}_i^{ord}$ the order is simply reversed. The first boundary vertex becomes the last, and the last boundary vertex becomes the first. In the following, it is assumed that both T_i's vertices and the vertices of the platelet boundary polygon are oriented counterclockwise.

Figure 5.2. illustrates different triangle platelets with platelet boundary polygons in counterclockwise order. Arrows on edges indicate the orientation.

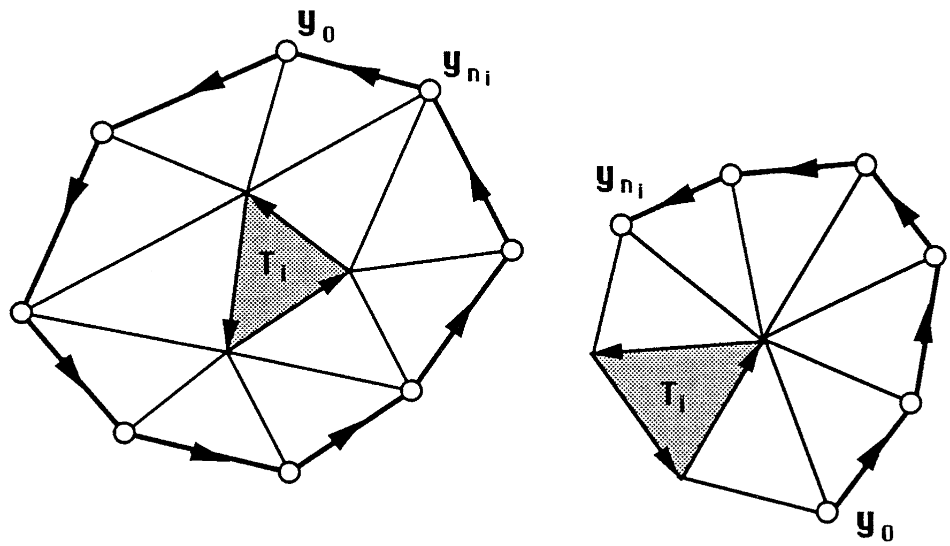

Fig. 5.2. Triangle platelet with continuous, acyclic corona and boundary vertex set.

Based on a half-plane test a criterion is introduced to decide, whether a triangle T_i in a triangulation can be removed or not. This test requires the following steps.

(i) Determine the plane equation of the plane P given by T_i.

(ii) Define an orthonormal coordinate system in P with T_i's centroid as origin and two arbitrary unit vectors in P.

(iii) Compute the distances of all points in the ordered boundary vertex set $\mathcal{B}_i^{ord}$ from P.

(iv) Project all points in $\mathcal{B}_i^{ord}$ onto P, and express the projected points with respect to the local coordinate system in P.

(v) Compute all line equations L_j in P determined by the projected directed line segments of the platelet boundary polygon.

(vi) Test, whether the centroid of T_i lies in the region obtained as the intersection of all half-planes $L_j > 0$.

Some steps are now discussed in detail. The outward unit normal vector of the plane P is given by

$$\mathbf{n} \;=\; (n^x, n^y, n^z)^T \;=\; \frac{\mathbf{d}_1 \times \mathbf{d}_2}{||\mathbf{d}_1 \times \mathbf{d}_2||}, \tag{5.9.}$$

where $\mathbf{d}_1 = \mathbf{a}_1 - \mathbf{a}_0$ and $\mathbf{d}_2 = \mathbf{a}_2 - \mathbf{a}_0$ are defined by T_i's vertices.

The plane equation for P is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{c}) \;=\; Ax + By + Cz + D \;=\; 0, \tag{5.10.}$$

where $\mathbf{c} = (x_0, y_0, z_0)^T = \frac{1}{3}\sum_{i=0}^{2} \mathbf{a}_i$ is T_i's centroid.

The unit basis vectors for the plane P can be chosen as

$$\mathbf{b}_1 \;=\; \frac{\mathbf{d}_1}{||\mathbf{d}_1||} \quad \text{and} \quad \mathbf{b}_2 \;=\; \mathbf{n} \times \mathbf{b}_1. \tag{5.11.}$$

As in chapter 4.2., the signed distances d_j, $j = 0...n_i$, of the platelet boundary points $\mathbf{y}_j = (x_j, y_j, z_j)^T$ are

$$d_j \;=\; Ax_j + By_j + Cz_j + D. \tag{5.12.}$$

Projecting the platelet boundary points onto P yields the points $\mathbf{y}_j^P$, where

$$\mathbf{y}_j^P \;=\; \mathbf{y}_j - d_j\; \mathbf{n}. \tag{5.13.}$$

Expressing the points $\mathbf{y}_j^P$ in P with respect to the two-dimensional coordinate system given by its origin $\mathbf{c}$ and the two basis vectors $\mathbf{b}_1$ and $\mathbf{b}_2$, one obtains

$$\mathbf{d}_j \;=\; (\mathbf{d}_j \cdot \mathbf{b}_1)\;\mathbf{b}_1 + (\mathbf{d}_j \cdot \mathbf{b}_2)\;\mathbf{b}_2, \qquad (5.14.)$$

where $\mathbf{d}_j = \mathbf{y}_j^P - \mathbf{c}$, $j = 0...n_i$. Therefore, the local coordinates $(u_j, v_j)^T$ of a point $\mathbf{y}_j^P$ with respect to the planar coordinate system are given by

$$\left(u_j, v_j\right)^T \;=\; \left(\mathbf{d}_j \cdot \mathbf{b}_1, \mathbf{d}_j \cdot \mathbf{b}_2\right)^T. \qquad (5.15.)$$

Projected onto P, the points $\mathbf{y}_j^P$ form an oriented polygon as well. The line equations of the single segments are expressed using the local planar coordinate system. The implicit line equation for the line $L_j(u,v)$ is given by

$$L_j\;(u,v) \;=\; -\;\triangle\, v_j\;(u - u_j) \;+\; \triangle u_j\;(v - v_j) \;=\; 0, \qquad (5.16.)$$

where $\triangle u_j = u_{(j+1)\bmod (n_i+1)} - u_j$ and $\triangle v_j = v_{(j+1)\bmod (n_i+1)} - v_j$, $j = 0...N$.

Now, the criterion is given to decide, whether the triangle T_i can be removed. If the centroid $\mathbf{c}$ (with local coordinates $(0,0)^T$) is on the “left,” “positive” side of all lines L_j the triangle can be removed.

Definition 5.9. The solution set of the $N+1$ linear inequalities

$$L_j\;(u,v) \;=\; -\;\triangle\, v_j\;(u - u_j) \;+\; \triangle u_j\;(v - v_j) \;>\; 0, \qquad (5.17.)$$

$j = 0...N$, is called the **feasible region** of the triangle platelet $\mathcal{TP}_i$ in the plane P.

Theorem 5.3. *The centroid* $\mathbf{c}$ *of a triangle* T_i *is in the feasible region defined by the inequalities in (5.17.) if the inequalities*

$$u_j\;\triangle v_j \;-\; v_j\;\triangle u_j \;>\; 0, \qquad (5.18.)$$

$j = 0...N$, *hold for all* j.

Proof. Assuming that the intersection of all half-planes defined by (5.17.) is not empty and inserting the local coordinates $(0,0)^T$ of the centroid $\mathbf{c}$ into (5.17.) proves the theorem.

q.e.d.

Remark 5.2. A triangle T_i can only be removed if it is surrounded by a continuous, acyclic corona, and its centroid passes the planar half-plane test.

In Figure 5.3., the half-plane test applied to the centroid of a triangle T_i passing the test is shown.

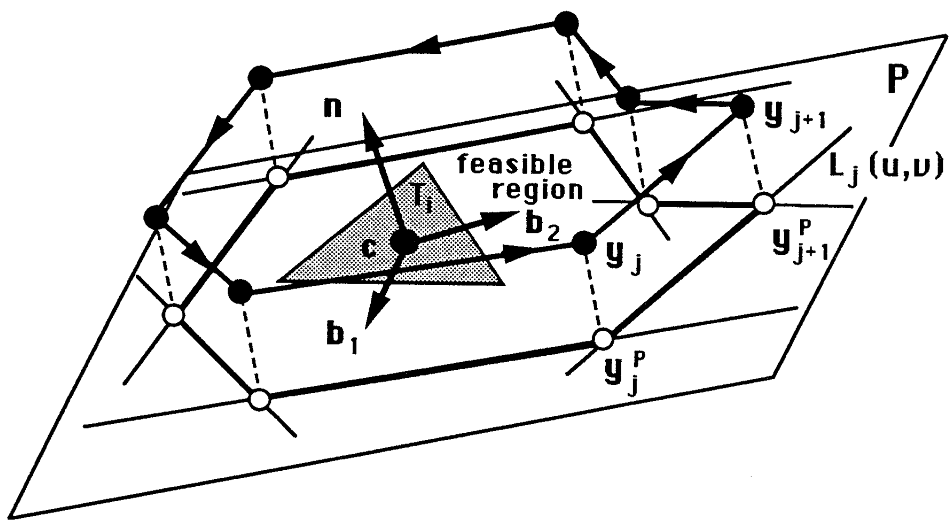

Fig. 5.3. Boundary vertex set and its projection onto plane P; triangle T_i passing the half-plane test.