Discussion 8: Solutions

ECS 17 (Winter 2024)

Patrice Koehl
koehl@cs.ucdavis.edu

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Induction

Exercise 1

Show that

$$\forall n \in \mathbb{N}, \sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$$

Let $P(n)$ be the predicate:

$$P(n): \quad \sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$$

We want to show that $P(n)$ is true for all natural numbers.

Let us define: $LHS(n) = \sum_{i=1}^{n} (-1)^i i^2$ and $RHS(n) = \frac{(-1)^n n(n+1)}{2}$. $P(n)$ is then $LHS(n) = RHS(n)$. To prove that $P(n)$ is true for all $n \geq 1$, we use a proof by induction.

- *Basis step*:

$$LHS(1) = (-1) \times 1^2 = 1 \qquad\qquad RHS(1) = \frac{(-1) \times 1 \times 2}{2} = 1$$

  Therefore $P(1)$ is true.

- *Induction step*: We suppose that $P(n)$ is true, with $1 \leq n$. We want to show that $P(n+1)$ is true.

$$
\begin{aligned}
LHS(n+1) &= \sum_{i=1}^{n+1}(-1)^i i^2 \\
&= \sum_{i=1}^{n}(-1)^i i^2 + (-1)^{n+1}(n+1)^2 \\
&= LHS(n) + (-1)^{n+1}(n+1)^2 \\
&= RHS(n) + (-1)^{n+1}(n+1)^2 \\
&= \frac{(-1)^n n(n+1)}{2} + (-1)^{n+1}(n+1)^2 \\
&= \frac{(-1)^n n(n+1) + 2(-1)^{n+1}(n+1)^2}{2} \\
&= \frac{(-1)^{n+1}(n+1)(2n+2-n)}{2} \\
&= \frac{(-1)^{n+1}(n+1)(n+2)}{2}
\end{aligned}
$$

and

$$RHS(n+1) = \frac{(-1)^{n+1}(n+1)(n+2)}{2}$$

Therefore $LHS(n+1) = RHS(n+1)$, which validates that $P(n+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n > 0$.

Exercise 2

Show that

$$\forall n \geq 4, \quad 2^n \leq n!$$

Let us define $LHS(n) = 2^n$ and $RHS(n) = n!$. Let $P(n)$ be the proposition: $LHS(n) \leq RHS(n)$. We want to show that $P(n)$ is true for all $n \geq 4$.

- *Basis step:* We show that $P(4)$ is true:

$$LHS(4) = 2^4 = 16$$
$$RHS(4) = 4! = 24$$

  Therefore $LHS(4) \leq RHS(4)$ and $P(4)$ is true.

- *Inductive step:* Let n be a positive integer greater or equal to 4 ($n \geq 4$), and let us suppose that $P(n)$ is true. We want to show that $P(n+1)$ is true.

$$LHS(n+1) = 2^{n+1} = 2LHS(n)$$

Since P(n) is true, we find:

$$LHS(n+1) \leq 2n!$$

Since $n \geq 4$, $2 \leq n+1$.
Therefore

$$\begin{aligned} LHS(n+1) &\leq (n+1) \times n! \\ LHS(n+1) &\leq (n+1)! \end{aligned}$$

Since $RHS(n+1) = (n+1)!$, we get $LHS(n+1) < RHS(n+1)$ which validates that $P(n+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 4$.

Exercise 3

Show that

$$\forall n \geq 1, \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

Let us define: $LHS(n) = \sum_{i=1}^{n} \frac{1}{(i)(i+1)}$ and $RHS(n) = \frac{n}{n+1}$. Let $P(n)$ be the proposition: $LHS(n) = RHS(n)$. We want to show that $P(n)$ is true for all $n > 0$. We use a proof by induction.

- *Basis step*:

$$LHS(1) = \frac{1}{1 \times 2} = \frac{1}{2} \qquad RHS(1) = \frac{1}{2}$$

  Therefore $LHS(1) = RHS(1)$, and $P(1)$ is true.

- *Induction step*: We suppose that $P(n)$ is true, with $1 \leq n$. We want to show that $P(n+1)$ is true.

$$
\begin{aligned}
LHS(n+1) &= \sum_{i=1}^{n+1} \frac{1}{i(i+1)} \\
&= \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} \\
&= LHS(n) + \frac{1}{(n+1)(n+2)} \\
&= RHS(n) + \frac{1}{(n+1)(n+2)} \\
&= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n(n+2)+1}{(n+1)(n+2)} \\
&= \frac{n^2+2n+1}{(n+1)(n+2)} \\
&= \frac{(n+1)^2}{(n+1)(n+2)} \\
&= \frac{n+1}{n+2}
\end{aligned}
$$

and

$$
RHS(n+1) = \frac{n+1}{n+2}
$$

Therefore $LHS(n+1) = RHS(n+1)$, which validates that $P(n+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all n.