

# Review session: Solutions

ECS 17 (Winter 2024)

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## 1 Simple propositions

For each proposition on the left, indicate if it is a tautology or not:

Table 1: Propositional logic

Proposition	Tautology (Yes/ No)
$(\neg(p \wedge q)) \leftrightarrow (\neg p \vee \neg q)$	Yes: this is one of DeMorgan's laws
$(\neg(p \wedge q)) \leftrightarrow (\neg p \wedge \neg q)$	No! contradicts DeMorgan's law
$(\neg(p \vee q)) \leftrightarrow (\neg p \wedge \neg q)$	Yes: this is the second DeMorgan's law
if $6^2 = 36$ then $2 = 3$	No: $p$ is true and $q$ is false: therefore $p \rightarrow q$ is false
if $6^2 = -1$ then $36 = -1$	Yes: $p$ is false and therefore $p \rightarrow q$ is always true.

## 2 Knights and Knaves

A very special island is inhabited only by Knights and Knaves. Knights always tell the truth, while Knaves always lie. You meet three inhabitants: Alex, John and Sally. Alex says, "If John is a Knight then Sally is a Knight". John says, "Alex is a Knight and Sally is a Knave". Can you find what Alex, John, and Sally are? Explain your answer.

Let us build the table for the possible options for Alex, John, and Sally. We then check the validity of the two statements, and finally check the consistency of the truth values for those statements with the nature of Alex and John.

Line	Alex	John	Sally	Alex says	John says	Compatibility
1	Knight	Knight	Knight	T	F	No: John would be a Knight who lies
2	Knight	Knight	Knave	F	T	No: Alex would be a Knight who lies
3	Knight	Knave	Knight	T	F	Yes
4	Knight	Knave	Knave	T	T	No, John would be a Knave who tells the truth
5	Knave	Knight	Knight	T	F	No, Alex would be a Knave who tells the truth
6	Knave	Knight	Knave	F	F	No, John would be a Knight who lies
7	Knave	Knave	Knight	T	F	No, Alex would be a Knave who tells the truth
8	Knave	Knave	Knave	T	F	No, Alex would be a Knave who tells the truth

Therefore Alex and Sally are Knights and John is a Knave.

### 3 Proofs: direct, indirect, and contradictions

#### 3.1 Different methods of proofs

Let  $n$  be an integer. Show that if  $3n^2 + 2n + 9$  is odd, then  $n$  is even using a direct, indirect, and proof by contradiction.

This is a problem of showing a conditional  $p \rightarrow q$  is true, where

$p$  :  $3n^2 + 2n + 9$  is odd

$q$  :  $n$  is even

We will use three different types of proof: direct, indirect, and proof by contradiction

a) Direct proof: we show directly that  $p \rightarrow q$  is true.

Hypothesis:  $p$  is true,  $3n^2 + 2n + 9$  is odd. Therefore there exists an integer  $k$  such that  $3n^2 + 2n + 9 = 2k + 1$ , i.e.  $3n^2 = 2k - 2n - 8$ . Noticing that  $3n^2 = n^2 + 2n^2$ , we get:  $n^2 = 2k - 2n - 8 - 2n^2 = 2(k - n - 4 - n^2)$ . Since both  $n$  and  $k$  are integers,  $k - n - 4 - n^2$  is an integer, which we call  $l$ . Then  $n^2 = 2l$ , i.e.  $n^2$  is even. We have seen in class that for all integers  $n$ ,  $n^2$  is even if and only if  $n$  is even. We can conclude that  $n$  is even:  $q$  is true. Therefore  $p \rightarrow q$  is true.

b) Indirect proof: Instead of showing directly that  $p \rightarrow q$  is true, we show that  $\neg q \rightarrow \neg p$  is true

Hypothesis:  $\neg q$  is true,  $n$  is odd.

Since  $n$  is odd, there exists an integer  $k$  such that  $n = 2k + 1$ . Therefore,  $3n^2 + 2n + 9 = 3(2k + 1)^2 + 2(2k + 1) + 9 = 12k^2 + 16k + 14 = 2(6k^2 + 8k + 7)$

Since  $6k^2 + 8k + 7$  is integer,  $3n^2 + 2n + 9$  is even, i.e.  $\neg p$  is true. Therefore  $\neg q \rightarrow \neg p$  is true, and  $p \rightarrow q$  is true.

c) Proof by contradiction: we suppose  $p \rightarrow q$  is false

Hypothesis:  $p \rightarrow q$  is false, i.e.  $p$  is true and  $\neg q$  is true, namely  $3n^2 + 2n + 9$  is odd and  $n$  is odd.

Since  $n$  is odd, there exists an integer  $k$  such that  $n = 2k + 1$ . Therefore,  $3n^2 + 2n + 9 = 3(2k + 1)^2 + 2(2k + 1) + 9 = 12k^2 + 16k + 14 = 2(6k^2 + 8k + 7)$

Since  $6k^2 + 8k + 7$  is integer,  $3n^2 + 2n + 9$  is even. But we have supposed that  $3n^2 + 2n + 9$  is odd. We have reached a contradiction. Therefore the hypothesis we made is false, therefore  $p \rightarrow q$  is true.

### 3.2 Proof by contradiction

Let  $n$  be a strictly positive integer. Show that  $\frac{2n+1}{2n+4}$  is not an integer

We use a proof by contradiction: We suppose that there exists an integer  $n$  such that  $\frac{2n+1}{2n+4}$  is an integer. Let us write the integer  $\frac{2n+1}{2n+4}$  as  $k$ . Then we have:

$$2n + 1 = k(2n + 4) = 2k(n + 2)$$

This would mean however that an odd number,  $2n + 1$ , is equal to an even number,  $2k(n + 2)$ . This is a contradiction. Therefore the hypothesis is wrong and the initial property, namely  $\frac{2n+1}{2n+4}$  is not an integer, is true.

### 3.3 Proof by contradiction

Let  $n$  be a strictly positive integer. Show that if  $\sqrt{n^2 + 1}$  is not an integer.

We use a proof by contradiction: We make the hypothesis that  $\sqrt{n^2 + 1}$  is an integer. Let us write this integer as  $k$ . Then we have:

$$\begin{aligned}\sqrt{n^2 + 1} &= k \\ n^2 + 1 &= k^2 \\ k^2 - n^2 &= 1 \\ (k - n)(k + n) &= 1\end{aligned}$$

Since  $k$  and  $n$  are supposed to be integers, there are only two possibilities:

- a)  $k - n = 1$  and  $k + n = 1$ , in which case  $k = 1$  and  $n = 0$ .
- b)  $k - n = -1$  and  $k + n = -1$ , in which case  $k = -1$  and  $n = 0$ .

In both cases, we have  $n = 0$ . However,  $n$  is set to be strictly positive. We have reached a contradiction, and therefore  $\sqrt{n^2 + 1}$  is not an integer.

## 4 Proofs by induction

## 4.1 Identity

- a) Show that  $1 + 3 + \dots + 2n - 1 = n^2$ , for all  $n \geq 1$ .

Let us define  $LHS(n) = 1 + 3 + \dots + 2n - 1$

and  $RHS(n) = n^2$

Let  $p(n) : LHS(n) = RHS(n)$

We want to show  $p(n)$  is true for all  $n \geq 1$

- a) Basis step: we want to show that  $p(1)$  is true

$$LHS(1) = 1$$

$$RHS(1) = 1^2 = 1$$

Since  $LHS(1) = RHS(1)$ ,  $p(1)$  is true

- b) Inductive Step

I want to show  $p(n) \rightarrow p(n+1)$  whenever  $n \geq 1$

Hypothesis:  $p(n)$  is true and  $LHS(n) = RHS(n)$

$$\begin{aligned} LHS(n+1) &= 1 + 3 + \dots + 2n - 1 + 2n + 1 \\ &= LHS(n) + 2n + 1 \\ &= RHS(n) + 2n + 1 \\ &= n^2 + 2n + 1 \\ &= (n+1)^2 \\ &= RHS(n+1) \end{aligned}$$

Therefore  $LHS(n+1) = RHS(n+1)$ , which validates that  $P(n+1)$  is true.

The principle of proof by mathematical induction allows us to conclude that  $P(n)$  is true for all  $n \geq 1$ .

- b) Show that  $\sum_{k=1}^n \frac{1}{4k^2 - 1} = \frac{n}{2n+1}$  for all integer  $n \geq 1$ .

Let us define  $LHS(n) = \sum_{k=1}^n \frac{1}{4k^2 - 1}$

and  $RHS(n) = \frac{n}{2n+1}$

Let  $p(n) : LHS(n) = RHS(n)$

We want to show  $p(n)$  is true for all  $n \geq 1$

- a) Basis step: we want to show that  $p(1)$  is true

$$LHS(1) = \frac{1}{3}$$

$$RHS(1) = \frac{1}{2 \times 1 + 1} = \frac{1}{3}$$

Since  $LHS(1) = RHS(1)$ ,  $p(1)$  is true

b) Inductive Step

I want to show  $p(n) \rightarrow p(n+1)$  whenever  $n \geq 1$

Hypothesis:  $p(k)$  is true and  $LHS(k) = RHS(k)$

$$\begin{aligned} LHS(n+1) &= \sum_{i=1}^{n+1} \frac{1}{4i^2 - 1} \\ &= LHS(n) + \frac{1}{4(n+1)^2 - 1} \\ &= RHS(n) + \frac{1}{4(n+1)^2 - 1} \\ &= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} \\ &= \frac{n(2n+3) + 1}{(2n+1)(2n+3)} \\ &= \frac{2n^2 + 3n + 1}{(2n+1)(2n+3)} \\ &= \frac{(2n+1)(n+1)}{(2n+1)(2n+3)} \\ &= \frac{n+1}{2n+3} \\ &= RHS(n+1) \end{aligned}$$

Therefore  $LHS(n+1) = RHS(n+1)$ , which validates that  $P(n+1)$  is true.

The principle of proof by mathematical induction allows us to conclude that  $P(n)$  is true for all  $n \geq 1$ .

## 4.2 Multiples

For the next two problems, we say that an integer  $n$  is a multiple of an integer  $m$  if and only if there exist an integer  $k$  such that  $n = km$ .

a) Show that  $(7^n - 2^n)$  is a multiple of 5 for all integer  $n \geq 1$ .

Let us define  $LHS(n) = 7^n - 2^n$

Let  $p(n) : LHS(n)$  is a multiple of 5.

We want to show  $p(n)$  is true for all  $n \geq 1$

a) Basis step: we want to show that  $p(1)$  is true

$$LHS(1) = 7 - 2 = 5$$

Since  $LHS(1) = 5 \times 1$ , and 1 is an integer,  $p(1)$  is true

b) Inductive Step

I want to show  $p(n) \rightarrow p(n+1)$  whenever  $n \geq 1$

$p(n)$  is true means there exists an integer  $m$  such that  $LHS(n) = 7^n - 2^n = 5m$ .

Note that:

$$\begin{aligned}LHS(n+1) &= 7^{n+1} - 2^{n+1} \\ &= 7 \times 7^n - 2 \times 2^n \\ &= 7 \times (5m + 2^n) - 2 \times 2^n \\ &= 5(7m) + 5 \times 2^n \\ &= 5(7m + 2^n)\end{aligned}$$

Since  $7m + 2^n$  is an integer,  $LHS(n+1)$  is a multiple of 5, which validates that  $P(n+1)$  is true.

The principle of proof by mathematical induction allows us to conclude that  $P(n)$  is true for all  $n \geq 1$ .

b) Show that  $[n(2n+1)(7n+1)]$  is a multiple of 6 for all integer  $n \geq 1$ .

Let us define  $LHS(n) = n(2n+1)(7n+1)$

Let  $p(n) : LHS(n)$  is a multiple of 6

We want to show  $p(n)$  is true for all  $n \geq 1$

a) Basis step:  $n=1$

$$LHS(1) = 1 \times (3) \times (8) = 24 = 6 \times 4$$

Since 4 is an integer,  $LHS(1)$  is a multiple of 6:  $p(1)$  is true

b) Inductive Step

I want to show  $p(n) \rightarrow p(n+1)$  whenever  $n \geq 1$

$p(n)$  is true means there exists an integer  $m$  such that  $LHS(n) = n(2n+1)(7n+1) = 6m$ .

Note that:

$$\begin{aligned}LHS(n+1) &= (n+1)(2n+3)(7n+8) \\ &= (2n^2 + 5n + 3)(7n + 8) \\ &= 14n^3 + 51n^2 + 61n + 24\end{aligned}$$

Note also that

$$\begin{aligned}LHS(n) &= (2n^2 + n)(7n + 1) \\ &= 14n^3 + 9n^2 + n\end{aligned}$$

Therefore:

$$\begin{aligned}LHS(n+1) &= LHS(n) + 42n^2 + 60n + 24 \\ &= 6m + 6(7n^2 + 10n + 4) \\ &= 6(m + 7n^2 + 10n + 4)\end{aligned}$$

As  $m + 7n^2 + 10n + 4$  is an integer (because both  $m$  and  $n$  are integers),  $LHS(n + 1)$  is a multiple of 6, which validates that  $P(n + 1)$  is true.

The principle of proof by mathematical induction allows us to conclude that  $P(n)$  is true for all  $n \geq 1$ .

### 4.3 Stamps: 1

Use induction to prove that any postage of  $n$  cents (with  $n \geq 30$ ) can be formed using only 6-cent and 7-cent stamps.

Let  $p(n)$  be the proposition that  $n$  cents can be made with only 6-cent and 7-cent stamps, when  $n$  is greater than or equal to 30.

Therefore there exists two positive integers  $a_n$  and  $b_n$  such that  $n = 6a_n + 7b_n$

a) Basis step: I want to prove that  $p(30)$  is true

30 can be composed of 5 times 6 plus 0 times 7:  $30 = 6 \times 5 + 7 \times 0$

We can set  $a_{30} = 5$  and  $b_{30} = 0$ . Both are positive integers. Therefore  $p(30)$  is true

b) Inductive Step

I want to show  $p(n) \rightarrow p(n + 1)$  whenever  $n \geq 30$

Hypothesis:  $p(n)$  is true and there exists two positive integers  $a_n$  and  $b_n$  such that  $n = 6a_n + 7b_n$

Then:

$$n + 1 = 6a_n + 7b_n + 1$$

Since 1 can be written as  $7 - 6$  we can write

$$n + 1 = 6a_n + 7b_n + 7 - 6 = 6(a_n - 1) + 7(b_n + 1)$$

Since  $b_n$  is greater than or equal to 0, then  $(b_n + 1)$  is also greater than 0

$(a_n - 1)$  is only positive if  $a_n$  is greater or equal to 1.

There are therefore two situations that we need to consider:  $a_n \geq 1$  and  $a_n = 0$ .

i)  $a_n \geq 1$

Then  $n + 1$  can be written as:

$$n + 1 = 6(a_n - 1) + 7(b_n + 1) \text{ where both } (a_n - 1) \text{ and } (b_n + 1) \text{ are positive.}$$

We can set  $a_{n+1} = a_n - 1$  and  $b_{n+1} = b_n + 1$ . In this case,  $p(n + 1)$  is true.

ii)  $a_n = 0$

$$n + 1 = 7b_n + 1$$

$$n + 1 = 7b_n + 36 - 35$$

$$n + 1 = 6 \times 6 + 7(b_n - 5)$$

$n + 1$  can be written as 6 times a positive integer 6 and 7 times  $(b_n - 5)$ .

Notice that  $n = 7b_n$ . Since  $n > 29$ ,  $7b_n > 29$ . Since  $b_n$  is an integer, we conclude that  $b_n \geq 5$ . Therefore  $(b_n - 5) \geq 0$ .

We can set  $a_{n+1} = 6$  and  $b_{n+1} = b_n - 5$ . In this case,  $p(n + 1)$  is true

In all cases, we have proven that  $p(n + 1)$  is true: the inductive step is true.

The principle of proof by mathematical induction allows us to conclude that  $p(n)$  is true for all  $n > 29$ .

#### 4.4 Stamps: 2

Use induction to prove that any postage of  $n$  cents (with  $n \geq 18$ ) can be formed using only 3-cent and 10-cent stamps.

Let  $p(n)$  be the proposition that  $n$  cents can be made with only 3-cent and 10-cent stamps, when  $n$  is greater than 18.

Therefore there exists two positive integers  $a_n$  and  $b_n$  such that  $n = 3a_n + 10b_n$

a) Basis step: I want to prove that  $p(18)$  is true

18 can be composed of 3 times 6 plus 0 times 10:  $18 = 3 \times 6 + 10 \times 0$

We can set  $a_{18} = 6$  and  $b_{18} = 0$ . Both are positive integers. Therefore  $p(18)$  is true

b) Inductive Step

I want to show  $p(n) \rightarrow p(n + 1)$  whenever  $n \geq 18$

Hypothesis:  $p(n)$  is true and there exists two positive integers  $a_n$  and  $b_n$  such that  $n = 3a_n + 10b_n$

Then:

$$n + 1 = 3a_n + 10b_n + 1$$

Since 1 can be written as  $10 - 9 = 10 - 3 \times 3$  we can write

$$n + 1 = 3a_n + 10b_n + 10 - 3 \times 3 = 3(a_n - 3) + 10(b_n + 1)$$

Since  $b_n$  is greater than or equal to 0, then  $(b_n + 1)$  is also greater than 0

$(a_n - 3)$  is only positive if  $a_n$  is greater or equal to 3.

There are therefore four situations that we need to consider:  $a_n \geq 3$ ,  $a_n = 2$ ,  $a_n = 1$ , and  $a_n = 0$ .

i)  $a_n \geq 3$

Then  $n + 1$  can be written as:

$$n + 1 = 3(a_n - 3) + 10(b_n + 1) \text{ where both } (a_n - 3) \text{ and } (b_n + 1) \text{ are positive.}$$

We can set  $a_{n+1} = a_n - 3$  and  $b_{n+1} = b_n + 1$ . In this case,  $p(n + 1)$  is true.

ii)  $a_n = 2$

$$n + 1 = 10b_n + 7$$

$$n + 1 = 10b_n + 27 - 20$$

$$n + 1 = 10(b_n - 2) + 3 \times 9$$

$n + 1$  can be written as 3 times a positive integer 9 and 10 times  $(b_n - 2)$ .

Notice that  $n = 10b_n + 6$ . Since  $n > 17$ ,  $10b_n + 6 > 17$ , and therefore  $10b_n > 11$ . Since  $b_n$  is an integer, we conclude that  $b_n \geq 2$ . Therefore  $(b_n - 2) \geq 0$ .

We can set  $a_{n+1} = 9$  and  $b_{n+1} = b_n - 2$ . In this case,  $p(n + 1)$  is true



iii)  $a_n = 1$

$$n + 1 = 10b_n + 4$$

Since  $4 = 24 - 20 = 3 \times 8 - 2 \times 10$  we can write

$$n + 1 = 10(b_n - 2) + 3 \times 8$$

$n + 1$  can be written as 3 times a positive integer 8 and 10 times  $(b_n - 2)$ .

Notice that  $n = 10b_n + 3$ . Since  $n > 17$ ,  $10b_n > 14$ . Since  $b_n$  is an integer, we conclude that  $b_n \geq 2$  and therefore  $b_n - 2 \geq 0$ .

We can set  $a_{n+1} = 8$  and  $b_{n+1} = b_n - 2$ . In this case,  $p(n + 1)$  is true

iv)  $a_n = 0$

$$n + 1 = 10b_n + 1$$

Since  $1 = 21 - 20 = 3 \times 7 - 2 \times 10$  we can write

$$n + 1 = 10(b_n - 2) + 3 \times 7$$

$n + 1$  can be written as 3 times a positive integer 7 and 10 times  $(b_n - 2)$ .

Notice that  $k = 10b_n$ . Since  $n > 17$ ,  $10b_n > 17$ . Since  $b_n$  is an integer, we conclude that  $b_n \geq 2$  and therefore  $b_n - 2 \geq 0$ .

We can set  $a_{n+1} = 7$  and  $b_{n+1} = b_n - 2$ . In this case,  $p(n + 1)$  is true

In all cases, we have proven that  $p(n + 1)$  is true: the inductive step is true.

The principle of proof by mathematical induction allows us to conclude that  $p(n)$  is true for all  $n > 17$ .

## 4.5 Other

Prove by induction that for all  $n \geq 1$ , there exist two strictly positive integers  $a_n$  and  $b_n$  such that  $(1 + \sqrt{2})^n = a_n + b_n\sqrt{2}$ .

Let  $p(n)$  be the proposition that there exist two strictly positive integers  $a_n$  and  $b_n$  such that  $(1 + \sqrt{2})^n = a_n + b_n\sqrt{2}$ .

We want to show  $p(n)$  is true for all  $n \geq 1$

a) Basis step: we want to show  $p(1)$  is true.

Note that  $(1 + \sqrt{2}) = 1 + 1 \times \sqrt{2}$ . Setting  $a_1 = 1$  and  $b_1 = 1$ , we have  $(1 + \sqrt{2}) = a_1 + b_1\sqrt{2}$

Therefore  $p(1)$  is true

b) Inductive Step

I want to show  $p(n) \rightarrow p(n + 1)$  whenever  $n \geq 1$

Hypothesis:  $p(n)$  is true and there exists two positive integers  $a_k$  and  $b_k$  such that  $(1 + \sqrt{2})^n = a_n + b_n\sqrt{2}$

Then,

$$\begin{aligned}(1 + \sqrt{2})^{n+1} &= (1 + \sqrt{2})^n(1 + \sqrt{2}) \\ &= (a_n + b_n\sqrt{2})(1 + \sqrt{2}) \\ &= a_n + 2b_n + (a_n + b_n)\sqrt{2}\end{aligned}$$

Let us set  $a_{n+1} = a_n + 2b_n$  and  $b_{n+1} = a_n + b_n$ . We note first that  $a_{n+1}$  and  $b_{n+1}$  are strictly positive integers. Second, we have:

$$(1 + \sqrt{2})^{n+1} = a_{n+1} + b_{n+1}\sqrt{2}$$

Therefore  $P(n + 1)$  is true.

The principle of proof by mathematical induction allows us to conclude that  $P(n)$  is true for all  $n \geq 1$ .

## 4.6 Fibonacci

Let  $f_n$  be the Fibonacci numbers. show that  $f_{n-1}f_{n+1} - f_n^2 = (-1)^n$ , for all  $n > 1$ .

Let me define  $LHS(n) = f_{n-1}f_{n+1} - f_n^2$

Let me define  $RHS(n) = (-1)^n$

Let  $P(n) : LHS(n) = RHS(n)$

I want to show  $p(n)$  is true for all  $n > 1$

a) Basis step: we want to show  $P(1)$  is true.

$$LHS(2) = f_1f_3 - f_2^2 = (1)(2) - (1)^2 = (2) - (1) = 1$$

$$RHS(2) = (-1)^2 = 1$$

Since  $LHS(2) = RHS(2)$ ,  $p(2)$  is true

b) Inductive Step

I want to show  $p(n) \rightarrow p(n + 1)$  whenever  $n > 1$

Hypothesis:  $p(n)$  is true therefore  $LHS(n) = RHS(n)$

$$LHS(n + 1) = f_n f_{n+2} - f_{n+1}^2$$

$$LHS(n + 1) = f_n(f_n + f_{n+1}) - f_{n+1}^2$$

$$LHS(n + 1) = f_n^2 + f_n f_{n+1} - f_{n+1}^2$$

$$LHS(n + 1) = f_n^2 + f_{n+1}(f_n - f_{n+1})$$

Since  $f_{n-1} + f_n = f_{n+1}$  then  $f_n - f_{n+1} = -f_{n-1}$

$$LHS(n + 1) = f_n^2 + f_{n+1}(-f_{n-1})$$

$$LHS(n + 1) = f_n^2 - f_{n+1}f_{n-1}$$

$$LHS(n + 1) = -LHS(n)$$

Since  $LHS(n) = RHS(n)$

$$LHS(n + 1) = -RHS(n) = (-1)^{n+1}$$

$$RHS(n + 1) = (-1)^{n+1}$$

Therefore  $LHS(n + 1) = RHS(n + 1)$ , which validates that  $P(n + 1)$  is true.

The principle of proof by mathematical induction allows us to conclude that  $P(n)$  is true for all  $n > 1$ .