Data, Logic, and Computing

ECS 17 (Winter 2022)

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Midterm 2: solutions

Exercise 1 *(1 question, 10 points)*

Let a and b be two real numbers, with $a \neq 0$ and $b \neq 0$. Use a proof by contradiction to show that if $ab > 0$, then $\frac{a}{b} + \frac{b}{a} \geq 2$.

Let:

p: $ab > 0$

q: $\frac{a}{b} + \frac{b}{a} \geq 2$

and let A be the proposition $p \rightarrow q$. We want to show that A is true. We use a proof by contradiction, i.e. we suppose that what we want to show is false, namely that A is not true, i.e. p is true AND q is false.

p is true: $ab > 0$. Similarly, as q is false, $\frac{a}{b} + \frac{b}{a} < 2$. As $ab > 0$, we can multiply this inequality by ab without changing its sense; we get:

$$a^2 + b^2 < 2ab$$

which gives

$$a^2 + b^2 - 2ab < 0$$

i.e.

$$(a-b)^2 < 0$$

However, $(a-b)^2$ is a square, and therefore $(a-b)^2 \geq 0$. we have reached a contradiction. The proposition A is therefore true.

Exercise 2 *2 questions, each 10 points; total 20 points)*

1) *Let a and b be two integers. Show that if $a + b\sqrt{2} = 0$, then $a = 0$ and $b = 0$.*

   Let:

   p: $a + b\sqrt{2} = 0$

q: $a = 0$ and $b = 0$

and let A be the proposition $p \to q$. We want to show that A is true. We use a proof by contradiction, i.e. we suppose that what we want to show is false, namely that A is not true, i.e. p is true AND q is false.

As q is false, either $a \neq 0$ or $b \neq 0$. As b may be 0, we look at the two corresponding cases:

a) $b = 0$. Since p is true, we get $a = 0$. But this is in contradiction with $\neg q$: we have reached a contradiction

b) $b \neq 0$. We can then divide by b in p and we obtain:

$$\sqrt{2} = \frac{-a}{b}$$

As $-a$ and b are integers, with $b \neq 0$, this suggests that $\sqrt{2}$ is rational... but we know that this is not true. We have reached a contradiction.

In all cases, we have reached a contradiction. The proposition A is therefore true.

2) *Let m, n, p, and q be four integers. Show that $m + n\sqrt{2} = p + q\sqrt{2}$, if and only if $m = p$ and $n = q$.*

Let:

P: $m + n\sqrt{2} = p + q\sqrt{2}$

Q: $m = n$ and $p = q$

and let A be the proposition $P \leftrightarrow Q$. We want to show that A is true. We prove both $P \to Q$ and $Q \to P$.

a) Lest us prove $P \to Q$. We use a direct proof.
We assume P is true, namely that

$$m + n\sqrt{2} = p + q\sqrt{2}$$

which we rewrite as:

$$(m - p) + (n - q)\sqrt{2} = 0$$

As $m - p$ and $n - q$ are both integers, based on part 1, we find that $m - p = 0$ and $n - q = 0$, namely $m = p$ and $n = q$. Therefore Q is true, and consequently $P \to Q$ is true.

b) Lest us prove $Q \to P$. We use a direct proof.
As Q is true, $m = p$ and $n = q$. Then $m + n\sqrt{2} = p + q\sqrt{2}$, namely P is true. Therefore $Q \to P$ is true.

We have shown that $P \to Q$ and $Q \to P$ are true. Therefore A is true.

Exercise 3 *(1 question, 10 points)*

Let x be a real number. Show that $|x-1| \leq x^2 - x + 1$, where $||$ is the absolute value.

Let:

P: $|x-1| \leq x^2 - x + 1$

where x is a real number. Let us define $A(x) = |x-1|$ and $B(x) = x^2 - x + 1$. To show that P is true, we use a proof by case:

a) $x \geq 1$. Then,

$$\begin{aligned} A(x) &= x - 1 \\ B(x) &= x^2 - x + 1 \end{aligned}$$

Therefore

$$\begin{aligned} A(x) - B(x) &= x - 1 - x^2 + x - 1 \\ &= -(x^2 - 2x + 1) - 1 \\ &= -(x-1)^2 - 1 \\ &< 0 \end{aligned}$$

Therefore $A(x) \leq B(x)$ and P is true.

b) $x < 1$. Then,

$$\begin{aligned} A(x) &= -x + 1 \\ B(x) &= x^2 - x + 1 \end{aligned}$$

Therefore

$$\begin{aligned} A(x) - B(x) &= -x + 1 - x^2 + x - 1 \\ &= -x^2 \\ &< 0 \end{aligned}$$

Therefore $A(x) \leq B(x)$ and P is true.

In al cases, P is true.

Exercise 4 *(1 question, 10 points)*

Let a and b be two integers. Show that if $(a^2+b^2)^2$ is even, then $a+b$ is even.

Let:

p: $(a^2+b^2)^2$ is even

q: $a+b$ is even

and let A be the proposition $p \rightarrow q$. We want to show that A is true. We use a direct proof.

We assume that $(a^2+b^2)^2$ is even. Let $n = a^2 + b^2$; n is an integer. The assumption is therefore that n^2 is even. We know then that n is even. Therefore $a^2 + b^2$ is even, but this leads to $a+b$ is even, using the theorems we can assume to be true. Therefore q is true, and A is true by direct proof.

Exercise 5 *(1 question, 10 points)*

Let a and b be two real numbers. Show that if $a \neq -1$ and $b \neq -1$, then $a + b + ab \neq -1$.

Let:

p: $a \neq -1$ and $b \neq -1$

q: $a + b + ab \neq -1$

and let A be the proposition $p \rightarrow q$. We want to show that A is true. We use an indirect proof. We need to define first: $\neg p$: $a = -1$ or $b = -1$

$\neg q$: $a + b + ab = -1$

We assume $\neg q$. Therefore $a + b + ab = -1$. Using the hint, we find $(a + 1)(b + 1) - 1 = -1$, i.e. $(a + 1)(b + 1) = 0$. This leads to $a = -1$ or $b = -1$, namely $\neg p$ is true. Therefore A is true by indirect proof.

Exercise 6 *(1 question, 10 points)*

Let a and b be two integers. Show that $a^2 - 4b \neq 2$.

Let:

P: $a^2 - 4b \neq 2$

where a and b are two integers. To show that P is true, we use a proof by contradiction, namely we assume P is false:

$$a^2 - 4b = 2$$

We get:

$$a^2 = 2 + 4b = 2(1 + 2b)$$

As $1 + 2b$ is an integer, a^2 is even, and therefore a is even. There exists an integer k such that $a = 2k$. Replacing above, we get:

$$\begin{aligned} (2k)^2 - 4b &= 2 \\ 4k^2 - 4b &= 2 \end{aligned}$$

After division by 2,

$$2k^2 - 2b = 1$$

$2k^2 - 2b = 2(k^2 - b)$ and since $k^2 - b$ is an integer, $2k^2 - 2b$ is even. However, 1 is odd: we have reached a contradiction. Therefore P is true.