Data, Logic, and Computing

ECS 17 (Winter 2024)

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Homework 7 - For 2/28/2024

Exercise 1 (5 points each; total 20)

Determine the truth values of the following statements; justify your answers:

a) $\forall n \in \mathbb{N}, (n+2) > n$

The statement is True. Let us prove it.

Let n be a natural number. Let us define $A = n + 2$ and $B = n$. We notice that $A - B = n + 2 - n = 2 > 0$. Therefore, $A > B$, i.e. $n + 2 > n$. As this is true for all n, the statement is true.

b) $\exists n \in \mathbb{N}, 2n = 3n$

The statement is False. Let us prove it.

Let us solve first $2n = 3n$ where n is an integer. We find $3n - 2n = 0$, therefore $n = 0$. Therefore, the equation $2n = 3n$ is only true for $n = 0$. However, 0 does not belong to $\mathbb{N}$. We can conclude that $\forall n \in \mathbb{N}, 2n \neq 3n$; the property is false.

c) $\forall n \in \mathbb{Z}, 3n \leq 4n$

The statement is False. Let us prove it.

Let n be an integer. $3n \leq 4n$ is equivalent to $0 \leq n$. This means that $\forall n < 0, 3n > 4n$. Therefore, we can find $n \in \mathbb{Z}$ such that $3n > 4n$ (for example $n = -1$). The statement is false.

d) $\exists x \in \mathbb{R}, x^4 < x^2$

The statement is True. Let us prove it.

Notice that the statement is based on existence: we only need to find one example. if $x = \frac{1}{2}$. $x^2 = \frac{1}{4}$ and $x^4 = \frac{1}{16}$, in which case $x^4 < x^2$.

Exercise 2 (10 points each; total 50 points

Show that the following statements are true.

a) *Let x be a real number. Prove that if x^3 is irrational, then x is irrational.*

Proof: Let x be a real number. We define the two statements: $P(x) : x^3$ is irrational, and $Q(x) : x$ is irrational. We want to show $P(x) \to Q(x)$. We will prove instead its contrapositive: $\neg Q(x) \to \neg P(x)$, where $\neg Q(x) : x$ is rational, and $\neg P(x) : x^3$ is rational.

Hypothesis: $\neg Q(x)$ is true, namely x is rational. By definition, there exists two integers a and b, with $b \neq 0$, such that $x = \frac{a}{b}$. Then,

$$x^3 = \frac{a^3}{b^3}$$

Since a is an integer, a^3 is an integer. Similarly, since b is a non-zero integer, b^3 is a non zero integer. Therefore x^3 is rational, which concludes the proof.

b) *Let x be a positive real number. Prove that if x is irrational, then $\sqrt{x}$ is irrational.*

Proof: Let x be a real number. We define the two statements: $P(x) : x$ is irrational, and $Q(x) : \sqrt{x}$ is irrational. We want to show $P(x) \to Q(x)$. We will prove instead its contrapositive: $\neg Q(x) \to \neg P(x)$, where $\neg Q(x) : \sqrt{x}$ is rational, and $\neg P(x) : x$ is rational.

Hypothesis: $\neg Q(x)$ is true, namely $\sqrt{x}$ is rational. By definition, there exists two integers a and b, with $b \neq 0$, such that $\sqrt{x} = \frac{a}{b}$. Then,

$$x = \frac{a^2}{b^2}$$

Since a is an integer, a^2 is an integer. Similarly, since b is a non-zero integer, b^2 is a non zero integer. Therefore x is rational, which concludes the proof.

c) Prove or disprove that if a and b are two rational numbers, then a^b is also a rational number.

The property is in fact not true. Let $a = 2$ and $b = \frac{1}{2}$. Then $a^b = 2^{\frac{1}{2}} = \sqrt{2}$; but we have shown in class that $\sqrt{2}$ is irrational.

d) *let n be a natural number. Show that n is even if and only if $3n + 8$ is even.*

Proof. Let n be a natural number and let $P(n)$ and $Q(n)$ be the propositions n is even, and $3n + 8$ is even, respectively. We will show that $P(n) \to Q(n)$ and $Q(n) \to P(n)$.

i) $P(n) \to Q(n)$
Hypothesis: n is even. By definition of even numbers, there exists and integer k such that $n = 2k$. Then,

$$3n + 8 = 6k + 8 = 2(3k + 4)$$

Since $3k + 4$ is an integer, $3n + 8$ can be written in the form $2k'$, where k' is an integer; therefore, $3n + 8$ is even.

ii) $Q(n) \to P(n)$
We will show instead its contrapositive, namely $\neg P(n) \to \neg Q(n)$, where $\neg P(n) : n$ is odd, and $\neg Q(n) : 3n + 8$ is odd.
Hypothesis: n is odd. By definition of even numbers, there exists and integer k such that $n = 2k + 1$. Then,

$$3n + 8 = 6k + 3 + 8 = 2(3k + 5) + 1$$

Since $3k + 5$ is an integer, $3n + 8$ can be written in the form $2k' + 1$, where k' is an integer; therefore, $3n + 8$ is odd.

e) *Prove that either* $4 \times 10^{769} + 22$ *or* $4 \times 10^{769} + 23$ *is not a perfect square. Is your proof constructive, or non-constructive?*

Let $n = 4 \times 10^{769} + 22$. The two numbers are n and $n + 1$.

Proof by contradiction: Let us suppose that both n and $n + 1$ are perfect squares:

$$\exists k \in \mathbb{Z}, k^2 = n$$
$$\exists l \in \mathbb{Z}, l^2 = n + 1$$

Then

$$\begin{aligned} l^2 &= k^2 + 1 \\ (l - k)(l + k) &= 1 \end{aligned}$$

Since l and k are integers, there are only two cases:

- $l - k = 1$ and $l + k = 1$, i.e. $l = 1$ and $k = 0$. Then we would have $k^2 = 0$, i.e. $n = 0$: contradiction
- $l - k = -1$ and $l + k = -1$, i.e. $l = -1$ and $k = 0$. Again, contradiction.

We can conclude that the proposition is true.

Exercise 3 (10 points)

Let n *be a natural number and let* $a_1, a_2, \ldots, a_n$ *be a set of* n *real numbers. Prove that at least one of these numbers is greater than, or equal to the average of these numbers. What kind of proof did you use?*

We use a proof by contradiction.

Suppose none of the real numbers a_1, a_2, ..., a_n is greater than or equal to the average of these numbers, denoted by $\overline{a}$.

By definition

$$\overline{a} = \frac{a_1 + a_2 + ... + a_n}{n}$$

Our hypothesis is that:

$$\begin{aligned} a_1 &< \overline{a} \\ a_2 &< \overline{a} \\ ... &< ... \\ a_n &< \overline{a} \end{aligned}$$

We sum up all these equations and get the following:

$$a_1 + a_2 + ... + a_n < n * \overline{a}$$

Replacing $\overline{a}$ in equation (9) by its value given in equation (4) we get:

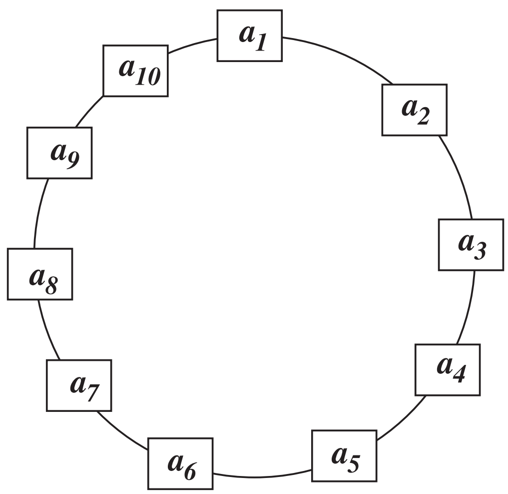
$$a_1 + a_2 + ... + a_n \quad < \quad a_1 + a_2 + ... + a_n$$

This is not possible: a number cannot be strictly smaller than itself: we have reached a contradiction. Therefore our hypothesis was wrong, and the original statement was correct.

Extra Credit (5 points)

Use Exercise 3 to show that if the first 10 strictly positive integers are placed around a circle, in any order, then there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 17.

Let a_1, a_2, ..., a_{10} be an arbitrary order of 10 positive integers from 1 to 10 being placed around a circle:



Since the ten numbers a correspond to the first 10 positive integers, we get:

$$a_1 + a_2 + ... + a_{10} \quad = \quad 1 + 2 + ... + 10 = 55 \tag{1}$$

Notice that the a_1, a_2, ..., a_{10} are not necessarily in the order 1, 2, ..., 10. They do include however the ten integers from 1 to 10: these is why the sum is 55

Let us now consider the different sums S_i of three consecutive sites around the circle. There are 10 such sums:

$$
\begin{aligned}
S_1 &= a_1 + a_2 + a_3 \\
S_2 &= a_2 + a_3 + a_4 \\
S_3 &= a_3 + a_4 + a_5 \\
S_4 &= a_4 + a_5 + a_6 \\
S_5 &= a_5 + a_6 + a_7 \\
S_6 &= a_6 + a_7 + a_8 \\
S_7 &= a_7 + a_8 + a_9 \\
S_8 &= a_8 + a_9 + a_{10} \\
S_9 &= a_9 + a_{10} + a_1 \\
S_{10} &= a_{10} + a_1 + a_2
\end{aligned}
$$

We do not know the values of the individual sums S_i; however, we can compute the sum of these numbers:

$$
\begin{aligned}
S_1 + S_2 + ... + S_{10} &= (a_1 + a_2 + a_3) + (a_2 + a_3 + a_4) + ... + (a_{10} + a_1 + a_2) \\
&= 3 * (a_1 + a_2 + ... + a_{10}) \\
&= 3 * 55 \\
&= 165
\end{aligned}
$$

The average of S_1, S_2, ..., S_{10} is therefore:

$$
\begin{aligned}
\overline{S} &= \frac{S_1 + S_2 + ... + S_{10}}{10} \\
&= \frac{165}{10} \\
&= 16.5
\end{aligned}
$$

Based on the conclusion of Exercise 3, at least one of S_1, S_2, ..., S_{10} is greater to or equal to $\overline{S}$, i.e., 16.5. Because S_1, S_2, ..., S_{10} are all integers, they cannot be equal to 16.5. Thus, at least one of S_1, S_2, ..., S_{10} is greater to or equal to 17.