Discussion 3/4

Exercise 1: Let F_n be the Fibonacci numbers. Show that for all natural numbers n, there exists an integer k such that

$$F_{4n} = 3k$$

Fibonacci numbers:

$$F_0 = 0$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2} \quad n \geq 2$$

Definition:

$P(n)$: F_{4n} is a multiple of 3.

I use a proof by induction to show

$P(n)$ is true for all $n \geq 1$

<u>Basis step:</u> is $P(1)$ true?

$$
\begin{aligned}
F_4 &= F_3 + F_2 \\
&= F_2 + F_1 + F_2 \\
&= F_1 + F_0 + F_1 + F_1 + F_0 \\
&= 3
\end{aligned}
$$

There exists an integer k ($k = 1$) such that

$F_4 = 3k$: F_4 is a multiple of 3.

$P(1)$ is true.

<u>Inductive step:</u> $P(n) \rightarrow P(n+1)$ is true for $n \geq 1$

I assume $P(n)$ is true.

There exists an integer k such that

$$F_{4n} = 3k$$

what about F_{4n+4}?

$$
\begin{aligned}
F_{4n+4} &= F_{4n+3} + F_{4n+2} \\
&= \underbrace{F_{4n+2}} + F_{4n+1} + F_{4n+1} + F_{4n} \\
&= F_{4n+1} + F_{4n} + 2F_{4n+1} + F_{4n}
\end{aligned}
$$

$$F_{4n+4} = 3F_{4n+1} + 2F_{4n}$$
$$= 3F_{4n+1} + 2 \times 3k$$
$$= 3\underbrace{(F_{4n+1} + 2k)}_{\text{integer}}$$

Therefore F_{4n+4} is a multiple of 3 : $P(n+1)$ is true.

The method of proof by induction allows me to conclude that $P(n)$ is true for all $n \geq 1$

<u>Exercise 2</u> Let F_n be the Fibonacci numbers. Show that

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

for all $n \geq 1$.

<u>Definitions</u>:
$$A(n) = F_{n-1}F_{n+1} - F_n^2$$
$$B(n) = (-1)^n$$
$$P(n): \ A(n) = B(n)$$

I want to show $P(n)$ is true for all $n \geq 1$.

I use a proof by induction.

Basis step: is $P(1)$ true?

$A(1) = F_0 F_2 - F_1^2 = -1$

$B(1) = (-1)^1 = -1$

$A(1) = B(1) \therefore P(1)$ is true.

Inductive step: $P(n) \rightarrow P(n+1)$ is true
$n \geqslant 1$

I assume $P(n)$ is true: $A(n) = B(n)$

$F_{n-1} F_{n+1} - F_n^2 = (-1)^n$

$A(n+1) = F_n F_{n+2} - F_{n+1}^2$

$= F_n (F_{n+1} + F_n) - F_{n+1}^2$

$= F_n F_{n+1} + F_n^2 - F_{n+1}^2$

$(F_{n+1} = F_n + F_{n-1} \rightarrow F_n = F_{n+1} - F_{n-1})$

$= (F_{n+1} - F_{n-1}) F_{n+1} + F_n^2 - F_{n+1}^2$

$= - F_{n-1} F_{n+1} + F_n^2 = -A(n) = -B(n) = -(-1)^n$

$B(n+1) = (-1)^{n+1}$

Therefore $A(n+1) = B(n+1)$: $P(n+1)$ is true.

The method of proof by induction allows us to conclude that $P(n)$ is true for all $n \geq 1$.

Exercise 3 Let a_n be the sequence:

$$a_1 = 1$$
$$a_n = a_{n-1} + 2n - 1 \quad n \geq 2$$

~~Show that~~ Find a closed form for a_n.

$$a_1 = 1$$
$$a_2 = a_1 + 2 \times 2 - 1 = 4 = 2^2$$
$$a_3 = a_2 + 2 \times 3 - 1 = 9 = 3^2$$
$$a_4 = a_3 + 2 \times 4 - 1 = 16 = 4^2$$

$P(n)$: $a_n = n^2$

Definitions:

$$A(n) = a_n$$
$$B(n) = n^2$$

$$P(n): \quad A(n) = B(n)$$

I want to show $P(n)$ is true for all $n \geq 1$, using induction.

<u>Basis step</u>: is $P(1)$ true?

$$A(1) = a_1 = 1$$
$$B(1) = 1^2 = 1$$

$P(1)$ is true.

<u>Inductive step</u>: $P(n) \to P(n+1)$ for $n \geq 1$

I assume $P(n)$ is true: $A(n) = B(n)$

$$a_n = n^2$$

$$\begin{aligned} A(n+1) = a_{n+1} &= a_n + 2(n+1) - 1 \\ &= n^2 + 2n + 2 - 1 \\ &= n^2 + 2n + 1 \\ &= (n+1)^2 \end{aligned}$$

$$B(n+1) = (n+1)^2$$

$A(n+1) = B(n+1)$: $P(n+1)$ is true.

The method of proof by induction allows me to conclude that $P(n)$ is true for all $n \geqslant 1$.