# Proofs 2/12

To prove $p \to q$

Show that if $n$ is even, then $n^3$ is even, when $n$ is an integer.

$p$: $n$ is even $\qquad \neg p$ $n$ is odd

$\Downarrow$ direct $\qquad \Uparrow$ indirect

$q$: $n^3$ is even $\qquad \neg q$ $n^3$ is odd.

We assume $p$ is true:

There exists an integer $k$ such that

$$n = 2k$$

$$n^3 = (2k)^3 = 8k^3 = 2\underbrace{(4k^3)}_{\text{integer}}.$$

$n^3$ is even. $q$ is true

Therefore $p \to q$ is true.

Show that if $n^3$ is odd, then $n$ is odd.

$p$: $n^3$ is odd $\qquad \neg p$: $n^3$ is even

$\downarrow \qquad\qquad\qquad\qquad \uparrow$

$q$: $n$ is odd $\qquad \neg q$: $n$ is even

Indirect proof:

I assume $\neg q$ is true.

There exists an integer $k$ such that

$n = 2k$

$$n^3 = (2k)^3 = 2\underbrace{(4k^3)}_{\text{integer}}$$

Therefore $\neg p$ is true.

Conclusion $\neg q \to \neg p$ is true, which is logically equivalent to $p \to q$ is true.

Example:

Show that if $n$ is an integer, then $n(n+1)$ is even.

$p$: $n$ is an integer

$q$: $n(n+1)$ is even

$\neg p$: $n$ is not an integer

$\neg q$: $n(n+1)$ is odd

Analysis: We only know that

$n$ is ~~even~~ an integer.

$p$: $n$ is ~~even~~ an integer

$p \Longleftrightarrow p_1 \vee p_2$

$p_1$: $n$ is even

$p_2$: $n$ is odd.

I need to prove:

$$(p_1 \vee p_2) \to q \Longleftrightarrow (p_1 \to q) \wedge (p_2 \to q)$$

We want to show that

if $n$ is even or $n$ is odd, then $n(n+1)$ is even

$p_1$ : $n$ is even

$p_2$ : $n$ is odd

$q$: $n(n+1)$ is even

Case 1 : $p_1 \to q$

We assume $p_1$ is true. There exists an integer $k$ such that $n = 2k$.

$$n(n+1) = 2k(2k+1) = 2\underbrace{\left(k(2k+1)\right)}_{\text{integer}}$$

Therefore $n(n+1)$ is even: $q$ is true.

Case 2 : $p_2 \to q$

We assume $p_2$ is true. There exists an integer $k$ such that $n = 2k+1$

$$n(n+1) = (2k+1)(2k+2)$$
$$= 2\underbrace{(2k+1)(k+1)}_{\text{integer}}$$

Therefore $n(n+1)$ is even: $q$ is true.

Example:

Let $m$ and $n$ be 2 integers.

Show that if $mn$ is odd, then $m$ is odd and $n$ is odd.

$p$: $mn$ is odd $\quad$ $\neg p$: $mn$ is even

$q$: $m$ is odd and $n$ is odd $\quad$ $\neg q$: $m$ is even OR $n$ is even.

We cannot use direct: we need to work with an indirect proof:

$$\neg q \longrightarrow \neg p$$

$m$ is even OR $n$ is even $\longrightarrow$ $mn$ is even.

Case 1 $\quad$ $m$ is even. There exists an integer $k$ such that $m = 2k$

$$m(n) = 2kn = 2\underbrace{(kn)}_{\text{integer}}$$

$mn$ is even: $\neg p$ is true.

Case 2 : $n$ is even

There exists an integer $k$ such that

$n = 2k$

$$mn = m \times 2k = 2\underbrace{(mk)}_{\text{integer}}$$

$mn$ is even : $\neg p$ is true.

therefore $\neg q \to \neg p$ is true,

and consequently, $p \to q$ is true.

Exercise:

Let $n$ be an integer. Show that if $n^2$ is even, then $n$ is even.

$p$ : $n^2$ is even  $\neg p$ : $n^2$ is odd

$q$ : $n$ is even  $\neg q$ : $n$ is odd

Direct proof:

We assume $p$ is true: $n^2$ is even.

there exist one integer $k$ such that $n^2 = 2k$

$$n^2 + n = n(n+1)$$

Since $n$ is an integer, $n(n+1)$ is even:

there exists an integer $l$ such that $n(n+1) = 2l$

$$2k + n = 2l$$

$$n = 2(\underbrace{l-k}_{\text{integer}})$$

$n$ is even.