Discussion 7: Solutions

ECS 20 (Winter 2019)

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Exercise 1

Let a, b, and c be three integers. Show that if $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.

We use a direct proof. Hypothesis: $a \mid bc$ and $\gcd(a, b) = 1$

Since $\gcd(a, b) = 1$, according to Bezout's identity, we know that there exists two integer numbers m and n such that

$$am + bn = 1$$

After multiplication by c:

$$acm + bcn = c$$

We know that $a \mid bc$. Therefore there exists an integer k such that $bc = ka$. Replacing in the equation above, we get:

$$\begin{aligned} acm + kan &= c \\ a(cm + kn) &= c \end{aligned}$$

i.e. $a \mid c$.

Exercise 2

Let n be a natural number. We call $s(n)$ the sum of its digits. We want to show that
if $s(3n) = s(n)$ then $9 \mid n$.

Proof. We use a direct proof.

Let n be a natural number. Since $3 \mid 3n$, we know that $3 \mid s(3n)$ (this is the divisibility property: a number is divisible by 3 if and only if 3 divides the sum of its digit).

The hypothesis is that $s(3n) = s(n)$. Therefore $3\ mids(n)$, i.e. $3 \mid n$ (from the same divisibility by 3 property).

As $3 \mid n$, there exists an integer k such that $n = 3k$. Then $3n = 9k$, i.e. $9 \mid 3n$. Applying the divisibility by 9 property (i.e. a number is divisible by 9 if and only if 9 divides the sum of its digits), we find that $9 \mid s(3n)$. Therefore $9 \mid s(n)$ and finally $9 \mid n$.

Exercise 3

Let a be a non-zero integer. Show that if $2 \nmid a$ and $3 \nmid a$, then $24 \mid (a^2 + 23)$.

Proof: we use a direct proof.

Let us consider the division of a by 6: there exists q and r such that $a = 6q + r$, with $0 \leq r < 6$. We note that $r \neq 0$ and $r \neq 2$ and $r \neq 4$, as otherwise we would have $2 \mid a$. Similarly, $r \neq 3$, as otherwise $3 \mid a$. There are only two cases left: $r = 1$ or $r = 5$. We consider the two cases separately:

1) $r = 1$

   $a = 6q + 1$, therefore $a^2 + 23 = (6q + 1)^2 + 23 = 36k^2 + 12k + 24 = 12k(3k + 1) + 24$. As k is an integer, we consider two cases:

k is even .

There exists an integer l such that $k = 2l$. Therefore, $a^2 + 23 = 24l(3k + 1) + 24 = 24(l(3k + 1) + 1)$, i.e. $24 \mid (a^2 + 23)$.

k is odd .

There exists an integer l such that $k = 2l+1$. Then $3k+1 = 6l+4 = 2(3l+2)$. Therefore $a^2 + 23 = 24k(3l + 2) + 24 = 24(k(3l + 2) + 1)$, i.e. $24 \mid (a^2 + 23)$.

We can conclude that when $a = 6q + 1$, $24 \mid (a^2 + 23)$.

2) $r = 5$

   $a = 6q + 5$, therefore $a^2 + 23 = (6q + 5)^2 + 23 = 36k^2 + 60k + 48 = 12k(3k + 5) + 48$. As k is an integer, we consider two cases:

k is even .

There exists an integer l such that $k = 2l$. Therefore, $a^2 + 23 = 24l(3k + 5) + 48 = 24(l(3k + 1) + 2)$, i.e. $24 \mid (a^2 + 23)$.

k is odd .

There exists an integer l such that $k = 2l+1$. Then $3k+5 = 6l+8 = 2(3l+4)$. Therefore $a^2 + 23 = 24k(3l + 4) + 48 = 24(k(3l + 4) + 2)$, i.e. $24 \mid (a^2 + 23)$.

We can conclude that when $a = 6q + 1$, $24 \mid (a^2 + 23)$.

The property is therefore true for all a such that $2 \nmid a$ and $3 \nmid a$.

Exercise 4

Since x, y, and z are natural numbers greater than 1, the number (xyz+1) is not divisible by either x, y or z, as xyz is a multiple of all of the three numbers, and (xyz+1)$\equiv 1 \mod x$, (xyz+1)$\equiv 1 \mod y$ and (xyz+1)$\equiv 1 \mod z$. Thus, we have proved by constructive proof that there exists at least one number greater than x, y, and z, which is not divisible by either of the three.