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ECS 20: Discrete Mathematics
Finals
December 9 2009

Notes:

1) Finals are open book, open notes.
2) You have two hours, no more: I will strictly enforce this.
3) You can answer directly on these sheets (preferred), or on loose paper.
4) Please write your name at the top right of each page you turn in!
5) Please, check your work!
6) There are 4 parts with a total possible number of points of 65. I will grade however over a total of 60, i.e. one question can be considered "extra credit". You choose!

Part I: logic (3 questions, each 5 points; total 15 points)

1) On a distant island, every inhabitant is either a Knight or Knave. Knights only tell the truth. Knaves only tell lies - everything said by a Knave is false. You meet three inhabitants: A, B and C. A says, "C is not a Knave". B says, "C and A are both Knights". C says, "A is a Knight or B is a Knave". Which, if any, are Knights? Which, if any, are Knaves?

Name:____________________________________

ID:______ ______________________________

2) Show that the proposition $\left[(r \vee p) \rightarrow (r \vee q)\right] \leftrightarrow \left[r \vee (p \rightarrow q)\right]$ is a tautology.

3) Let us play a logical game. You find yourself in front of three rooms whose doors are closed. One of these rooms contains a Lady, another a Tiger and the third room in empty. There is one sign on each door; you are told that the sign on the door of the room containing the Lady is true, the sign on the door of the room with the Tiger is false, and the sign on the door of the empty room could be either true or false. Here are the signs:

I	II	III
Room III is empty	The tiger is in room I	This room is empty

Which room contains the Lady, which room contains the Tiger, and which room is empty? Justify your answer

*Name:*____________________________________

*ID:*______ ______________________________

Part II: proofs and number theory (3 questions; each 5 points; total 15 points)

1) Prove or disprove that if p is prime, $3p+1$ is prime.

2) Show that the sum of any three consecutive perfect cubes is divisible by 9 (***Note***: a perfect cube is a number that can be written in the form n^3 where n is an integer. The three numbers $(n-1)^3$, n^3 and $(n+1)^3$ are three consecutive perfect cubes. (***Hint***: Start by showing that $n^3 + 2n \equiv 0[3]$ for all integer n)

*Name:*___________________________________
*ID:*______ ____________________________

3) *n* is a positive integer. Prove that $12 \mid (n^2-1)$ if gcd($n,6$) = *1* (***Hint:*** *write n=6k+l, and show that if gcd(n,6)=1, then l = 1 or l = 5).*

Part III. Proof by induction (4 questions; each 5 points; total 20 points)

1) Show by induction that $\sum_{i=1}^{n} \frac{2}{3^i} = 1 - \frac{1}{3^n}$ for all n≥1

Name:___________________________________
ID:______ ____________________________

2) Prove by induction that every number greater that 7 is the sum of a nonnegative integer multiple of 3 and a nonnegative integer multiple of 5.

3) Prove by induction that $2^{n+1} > n^2 + 1$ for all $n \geq 2$

*Name:*___________________________________
*ID:*______ ____________________________

4) Let f_n be the n-th Fibonacci number (note: Fibonacci numbers satisfy $f_0=0, f_1=1$ and $f_n + f_{n+1} = f_{n+2}$).

Prove by induction that for all $n\geq 1$, f_{3n} is even.

Part IV. Counting. Pigeonhole principle. (4 problems; each 5 points; total 20 points)

1) Prove that if *6* distinct numbers are selected from $\{1,2,\ldots,9,10\}$, then there will be at least two that are consecutive.

*Name:*___________________________________
*ID:*______ ____________________________

2) If 5 points are selected at random from the interior of the unit circle, then there are 2 points whose distance is less than $\sqrt{2}$

3) Given any five integers, there will be three for which the sum of the squares of those integers is divisible by 3. *(**Hint**: consider the Pigeonhole Principle, with two "boxes", "divisible by 3", and "not divisible by 3")*

Name:____________________________________
ID:______ _____________________________

4) A combination lock has three numbers in the combination, each in the range 1 to 40.
   a. How many different combinations are there?

   b. How many of the combinations have no duplicate numbers?

   c. How many of the combinations have exactly two of the three numbers matching?