ECS20
Homework 8: Number theory and Summations
Due March 5, 2019

Number theory:

Exercise 1 ***(10 points)***

Let a, b and n be three positive integers with $gcd(a,n) = 1$ and $gcd(b,n) = 1$. Show that $gcd(ab,n) = 1$

Exercise 2 ***(10 points)***

Prove that there are no solutions in integers x and y to the equation $2x^2+5y^2=14$. (*Hint:* consider this equation modulo 5)

Exercise 3 ***(10 points each; 20 points total)***

Use Fermat's little theorem to evaluate:

(i) 2^{302} mod 7
(ii) 5^{123} mod 61

Exercise 4 ***(10 points)***

Let n be an integer. Show that if $n>3$ then n, $n+2$ and $n+4$ cannot all be prime

Sequence, Summation:

Exercise 5 ***(5 points each; total: 20 points)***

Find the value of each of these sums (* means multiply):

a) $\sum_{j=0}^{8}\left(1+(-1)^j\right)$ b) $\sum_{j=0}^{8}\left(3^j-2^j\right)$

c) $\sum_{j=0}^{8}\left(2*3^j+3*2^j\right)$ d) $\sum_{j=0}^{8}\left(2^{j+1}-2^j\right)$

Exercise 6 ***(10 points)***

Using the identity $\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$, compute $\sum_{k=1}^{n}\frac{1}{k(k+1)}$, where n is a natural number

Exercise 7 ***(10 points)***

Without using mathematical induction, show that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all natural numbers $n \geq 1$. (hint: assume that you know that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, and compute $\sum_{i=1}^{n} (i+1)^4$ in two different ways)

Exercise 8 ***(10 points)***

Without using mathematical induction, prove that:

$\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$ for all natural numbers $n \geq 1$.

Extra credit ***(3 points)***

Let a and b be two natural numbers.

a) Show that if $gcd(a,b) = 1$ then $gcd(a+b, ab) = 1$
b) Show that if $gcd(a,b) = 1$ then $gcd(a^2+b^2, ab) = 1$