Homework 6 Solutions

ECS 20 (Winter 2019)

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February 13, 2019

Exercise 1

a) Show that $2x - 10$ is $\Theta(x)$.

One option is to prove that $2x - 10$ is both $\mathcal{O}(x)$ and $\Omega(x)$. In this simple case however, we directly "squeeze" $2x - 10$ between two functions that are of order x. First, let us notice that $\forall x \in \mathbb{R}, 2x - 10 < 2x$.

Second, we note that if $x > 10$, then $x - 10 > 0$ and therefore $2x - 10 > x$

Summarizing: for $x > 10$, $x < 2x - 10 < 2x$. Therefore $2x - 10$ is $\Theta(x)$.

b) Show that $4x^2 + 8x - 6$ is $\Theta(x^2)$.

Again, one option is to prove that $4x^2 + 8x - 6$ is both $\mathcal{O}(x^2)$ and $\Omega(x^2)$. In this simple case however, we directly "squeeze" $4x^2 + 8x - 6$ between two functions that are of order x^2.

We note first that when $8x > 6$, then $8x - 6 > 0$, and therefore $4x^2 + 8x - 6 > 4x^2$.

Second, we note that when $1 < x$, $x < x^2$, and therefore $8x < 8x^2$. We also have $-6 < x^2$. This leads to $4x^2 + 8x - 6 < 13x^2$ when $x > 1$.

Summarizing: for $x > 1$, $4x^2 < 4x^2 + 8x - 6 < 13x^2$. Therefore $4x^2 + 8x - 6$ is $\Theta(x^2)$.

c) Show that $\lfloor x + \frac{2}{7} \rfloor$ is $\Theta(x)$.

Again, we will "squeeze" $\lfloor x + \frac{2}{7} \rfloor$ between two functions that are of order x.

By definition of the function floor, $\lfloor x + \frac{2}{7} \rfloor \leq x + \frac{2}{7}$. If $\frac{2}{7} < x$, this leads to $\lfloor x + \frac{2}{7} \rfloor < 2x$.

Similarly, $x + \frac{2}{7} < \lfloor x + \frac{2}{7} \rfloor + 1$, therefore $x - \frac{5}{7} < \lfloor x + \frac{2}{7} \rfloor$.

If $x > 1$, then $-x < -1$; multiplying by $\frac{5}{7}$, $-\frac{5x}{7} < -\frac{5}{7}$, and adding x, we get $\frac{2x}{7} < x - \frac{5}{7}$, therefore $\frac{2x}{5} < \lfloor x + \frac{2}{7} \rfloor$.

Summarizing: for $x > 1$, $\frac{2x}{5} < \lfloor x + \frac{2}{7} \rfloor < 2x$. Therefore $\lfloor x + \frac{2}{7} \rfloor$ is $\Theta(x)$.

d) Show that $\log_4(x)$ is $\Theta \log_7(x)$.

Notice first that $\log_4(x) = \log_7(4) \times \log_7(x)$. Since $\log_4(x)$ and $\log_7(x)$ only differ by a (positive) constant, there are of the same order. Hence $\log_4(x)$ is $\Theta(\log_7(x))$.

Exercise 2

Show that x^2 is $\mathcal{O}(x^4)$ but that x^4 is not $\mathcal{O}(x^3)$.

a) Let us show that x^2 is $\mathcal{O}(x^4)$

   Let us assume that $1 < x$. Since $x > 0$, we can multiply this inequality by x: $x < x^2$, again: $x^2 < x^3$ and finally $x^3 < x^4$. As $x^2 < x^3$ and $x^3 < x^4$, we have $x^2 < x^4$.

   We have shown that there exists k ($k = 1$), and there exists C ($C = 1$), such that if $x > k$, then $x^2 < Cx^4$: we can conclude that x^2 is $\mathcal{O}(x^4)$.

b) Let us show that x^4 is not $\mathcal{O}(x^2)$.

   We use a proof by contradiction: let us suppose that the proposition is true, i.e. that x^4 is $\mathcal{O}(x^2)$. By definition of $\mathcal{O}$, this means that:

   $\exists k \in \mathbb{R}, \exists C \in \mathbb{R}$ if $x > k$ then $|x^4| < C|x^2|$.

   Let $D = \max\{2, k, C\}$. Therefore $D > 1$, $D \geq k$, and $D \geq C$.

   Let x be a real number with $x > D$. Since $D > 1$, $x > 1$ and therefore $x^2 > x > D$.

   Since $D \geq k$, we have $x^4 < Cx^2 < Dx^2$. Since $x > 0$, we can divide this inequality by x^2: we get $x^2 < D$. We have shown that $x^2 > D$ and $x^2 < D$: we have reached a contradiction. Therefore, the hypothesis, x^4 is $\mathcal{O}(x^2)$, is false. We can conclude that x^4 is not $\mathcal{O}(x^2)$.

Exercise 3

Let a, and b be two strictly positive integers and let x be a real number.. Show that:

$$\left\lfloor \frac{\left\lfloor \frac{x}{a} \right\rfloor}{b} \right\rfloor = \left\lfloor \frac{x}{ab} \right\rfloor$$

Let us define $k = \left\lfloor \frac{x}{a} \right\rfloor$ and $m = \left\lfloor \frac{x}{ab} \right\rfloor$. By definition of floor, we have the two properties:
$k \leq \frac{x}{a} < k + 1$
and
$m \leq \frac{x}{ab} < m + 1$
Let us multiply the second inequalities by b:
$bm \leq \frac{x}{a} < b(m + 1)$
We notice that:
$k \leq \frac{x}{a}$ and $\frac{x}{a} < b(m + 1)$; therefore $k < b(m + 1)$.
$k \leq \frac{x}{a}$ and $bm \leq \frac{x}{a}$. Therefore k and bm are two integers smaller than $\frac{x}{a}$. By definition of floor, k is the largest integer smaller that $\frac{x}{a}$. Therefore $bm \leq k$.

Combining those two inequalities, we get $bm \leq k < b(m+1)$. After division by b, $m < \frac{k}{b} < m+1$. Therefore m is the floor of $\frac{k}{b}$. Replacing m and k by their values, we get:

$$m = \left\lfloor \frac{x}{ab} \right\rfloor = \left\lfloor \frac{k}{b} \right\rfloor = \left\lfloor \frac{\left\lfloor \frac{x}{a} \right\rfloor}{b} \right\rfloor$$

The property is therefore true.

Exercise 4

Let x be a positive real number. Solve $\lfloor x\lfloor x\rfloor\rfloor = 5$.

Let $A = \lfloor x\lfloor x\rfloor\rfloor$.
Since $x \geq 0$, we do not need to worry about x being negative.
We notice first that if $x \geq 3$, then $\lfloor x\rfloor \geq 3$, and $x\lfloor x\rfloor \geq 9$, therefore $A \geq 9$.
Therefore possible solutions for x are between 0 and 3, 3 not included. We look at three cases:

a) $0 \leq x < 1$

In this case, $\lfloor x\rfloor = 0$ and $A = 0$. There are no solutions in this interval.

b) $1 \leq x < 2$

In this case, $\lfloor x\rfloor = 1$ and $A = \lfloor x\rfloor = 1$. There are no solutions in this interval.

c) $2 \leq x < 3$

In this case, $\lfloor x\rfloor = 2$ and $A = \lfloor 2x\rfloor$. Since $2 \leq x < 3$, $4 \leq 2x < 6$. We distinguish two cases:

i) $4 \leq 2x < 5$, namely $2 \leq x < 2.5$. Then $A = \lfloor 2x\rfloor = 4$; there are no solutions in this interval.

ii) $5 \leq 2x < 6$, namely $2.5 \leq x < 3$. Then $A = \lfloor 2x\rfloor = 5$; all values of x in this interval are solutions.

In conclusion, all values of $x \in [2.5, 3[$ are solutions of the equation.

Exercise 5

Let n be a natural number. Show that if n is a perfect square, then $2n$ is not a perfect square.

We will do a proof by contradiction. The property is of the form $P : p \to q$, where p is "n is a perfect square" and q is "$2n$ is not a perfect square". Assuming P is false is equivalent to assuming that p is true AND q is false. Therefore:
p is true: there exists an integer k such that $n = k^2$
q is false: there exists an integer l such that $2n = l^2$.
Since $n > 0$, $k \neq 0$ and $l \neq 0$. Replacing n with k^2 in the second equality, we get,
$2k^2 = l^2$. Taking the square root, we get $\sqrt{2}k = l$.... but this would say that $\sqrt{2}$ is rational. We have reached a contradiction: the property $\neg P$ is therefore false, and then P is true.

Extra Credit

Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy:
$\forall (x, y) \in \mathbb{R}^2, f(x)f(y) + f(x + y) = xy$

As the property is true for all pairs of real number, it is true for $(x, y) = (0, 0)$. Therefore:
$f(0)^2 + f(0) = 0$
from which we deduce that $f(0) = 0$ or $f(0) = -1$.

a) f(0) = 0

Applying the property to $(x, y) = (a, 0)$, where a is a real number. we get $f(a) = 0$ for all a, therefore f is the null function.

b) $f(0) = -1$

Applying the property to $(x, y) = (1, -1)$, we get : $f(1)f(-1)+f(0) = -1$, i.e. $f(1)f(-1) = 0$, i.e. $f(1) = 0$ or $f(-1) = 0$.

i) $f(1) = 0$.
We apply the property to $(x, y) = (a-1, 1)$, we get: $f(a-1)f(1)+f(a) = a-1$, therefore $f(a) = a - 1$

ii) $f(-1) = 0$
We apply the property to $(x, y) = (a + 1, -1)$, we get: $f(a + 1)f(-1) + f(a) = -a - 1$, therefore $f(a) = -a - 1$

We have found that if f satisfies the property, then f is one of the three following functions: $f_1(x) = 0$, $f_2(x) = x - 1$ and $f_3(x) = -x - 1$. We note however that $f_1(x)$ does not satisfy the property: let (x, y) be two real numbers; $f(x)f(y) + f(x+y) = 0$, while $xy = 0$ if and only if $x = 0$ or $y = 0$. For the other two functions,
$f_2(x)f_2(y) + f_2(x + y) = (x - 1)(y - 1) + x + y - 1 = xy$
and
$f_3(x)f_3(y) + f_3(x + y) = (x + 1)(y + 1) - x - y - 1 = xy$

Therefore f_2 and f_3 satisfy the property. They are the only solutions.