Homework 3: Solutions

ECS 20 (Winter 2019)

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January 20, 2019

Exercise 1 *(5 points)*

Let a, b, and c be three propositions. Show that this implication is a tautology, using a truth table:

$$(a \vee b) \wedge (\neg a \vee c) \rightarrow (b \vee c)$$

Let $A = (a \vee b) \wedge (\neg a \vee c) \rightarrow (b \vee c)$. The truth table for the values of A is:

a	b	c	$a \vee b$	$\neg a \vee c$	$(a \vee b) \wedge (\neg a \vee c)$	$b \vee c$	A
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T
T	F	T	T	T	T	T	T
T	F	F	T	F	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	F	T	T
F	F	F	F	T	F	F	T

A is always true, independent of the values of a, b, and c: it is a tautology.

Exercise 2 *(5 points)*

Let p, q, and r be three propositions. Show that $(p \vee q) \rightarrow r$ and $(p \rightarrow r) \vee (q \rightarrow r)$ are not logically equivalent.

Let $A = (p \vee q) \rightarrow r$ and $B = (p \rightarrow r) \vee (q \rightarrow r)$. Let us compare the truth values of A and B.

p	q	r	$p \vee q$	A	$p \rightarrow r$	$p \rightarrow r$	B
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

A and B do not always share the same truth values: they are not logically equivalent.

Exercise 3 *(5 points each; total 20 points)*

Determine the truth values of the following statements; justify your answers:

a) $\forall n \in \mathbb{N}, n < (n+2)$

The statement is True. Let us prove it.

Let n be a natural number. Let us define $A = n$ and $B = n + 2$. We notice that $A - B = n - (n+2) = -2 < 0$. Therefore, $A < B$, i.e. $n < (n+2)$. As this is true for all n, the statement is true.

b) $\exists n \in \mathbb{N}, 2n = 3n$

The statement is False. Let us prove it.

Let us solve first $2n = 3n$ where n is an integer. We find $2n - 3n = 0$, i.e. $n = 0$. Therefore, the equation $2n = 3n$ is only true for $n = 0$. However, 0 does not belong to $\mathbb{N}$. We can conclude that $\forall n \in \mathbb{N}, 2n \neq 3n$; the property is false.

c) $\forall n \in \mathbb{Z}, 3n \leq 4n$

The statement is False. Let us prove it.

Let n be an integer. $3n \leq 4n$ is equivalent to $0 \leq n$. This means that $\forall n < 0, 3n > 4n$. Therefore, we can find $n \in \mathbb{Z}$ such that $3n > 4n$ (for example $n = -1$). The statement is false.

d) $\exists x \in \mathbb{R}, x^3 < x^2$

The statement is True. Let us prove it.

Notice that the statement is based on existence: we only need to find one example. if $x = -1$. $x^2 = 1$ and $x^3 = -1$, in which case $x^3 < x^2$.

Exercise 4 *(5 points each; total 25 points)*

Show that the following statements are true.

a) Let x be a real number. Prove that if x^2 is irrational, then x is irrational.

Proof: Let x be a real number. We define the two statements: $P(x) : x^2$ is irrational, and $Q(x) : x$ is irrational. We want to show $P(x) \to Q(x)$. We will prove instead its contrapositive: $\neg Q(x) \to \neg P(x)$, where $\neg Q(x) : x$ is rational, and $\neg P(x) : x^2$ is rational.

Hypothesis: $\neg Q(x)$ is true, namely x is rational. By definition, there exists two integers a and b, with $b \neq 0$, such that $x = \frac{a}{b}$. Then,

$$x^2 = \frac{a^2}{b^2}$$

Since a is an integer, a^2 is an integer. Similarly, since b is a non-zero integer, b^2 is a non zero integer. Therefore x^2 is rational, which concludes the proof.

b) Let x be a positive real number. Prove that if x is irrational, then $\sqrt{x}$ is irrational.

Proof: Let x be a real number. We define the two statements: $P(x) : x$ is irrational, and $Q(x) : \sqrt{x}$ is irrational. We want to show $P(x) \to Q(x)$. We will prove instead its contrapositive: $\neg Q(x) \to \neg P(x)$, where $\neg Q(x) : \sqrt{x}$ is rational, and $\neg P(x) : x$ is rational.

Hypothesis: $\neg Q(x)$ is true, namely $\sqrt{x}$ is rational. By definition, there exists two integers a and b, with $b \neq 0$, such that $\sqrt{x} = \frac{a}{b}$. Then,

$$x = \frac{a^2}{b^2}$$

Since a is an integer, a^2 is an integer. Similarly, since b is a non-zero integer, b^2 is a non zero integer. Therefore x is rational, which concludes the proof.

c) Prove or disprove that if a and b are two rational numbers, then a^b is also a rational number.

The property is in fact not true. Let $a = 2$ and $b = \frac{1}{2}$. Then $a^b = 2^{\frac{1}{2}} = \sqrt{2}$; but we have shown in class that $\sqrt{2}$ is irrational.

d) let n be a natural number. Show that n is even if and only if $5n + 12$ is even.

Proof. Let n be a natural number and let $P(n)$ and $Q(n)$ be the propositions n is even, and $5n + 12$ is even, respectively. We will show that $P(n) \to Q(n)$ and $Q(n) \to P(n)$.

i) $P(n) \to Q(n)$
Hypothesis: n is even. By definition of even numbers, there exists and integer k such that $n = 2k$. Then,

$$5n + 12 = 10k + 12 = 2(5k + 6)$$

Since $5k + 6$ is an integer, $5n + 12$ can be written in the form $2k'$, where k' is an integer; therefore, $5n + 12$ is even.

ii) $Q(n) \to P(n)$
We will show instead its contrapositive, namely $\neg P(n) \to \neg Q(n)$, where $\neg P(n) : n$ is odd, and $\neg Q(n) : 5n + 12$ is odd.
Hypothesis: n is odd. By definition of even numbers, there exists and integer k such that $n = 2k + 1$. Then,

$$5n + 12 = 10k + 5 + 12 = 2(5k + 8) + 1$$

Since $5k + 8$ is an integer, $5n + 12$ can be written in the form $2k' + 1$, where k' is an integer; therefore, $5n + 12$ is odd.

e) Prove that either $4 \times 10^{769} + 22$ or $4 \times 10^{769} + 23$ is not a perfect square. Is your prove constructive, or non-constructive?

Let $n = 4 \times 10^{769} + 22$. The two numbers are n and $n+1$.

Proof by contradiction: Let us suppose that both n and $n+1$ are perfect squares:

$$\exists k \in \mathbb{Z}, k^2 = n$$
$$\exists l \in \mathbb{Z}, l^2 = n+1$$

Then

$$\begin{aligned} l^2 &= k^2 + 1 \\ (l-k)(l+k) &= 1 \end{aligned}$$

Since l and k are integers, there are only two cases:

- $l - k = 1$ and $l + k = 1$, i.e. $l = 1$ and $k = 0$. Then we would have $k^2 = 0$, i.e. $n = 0$: contradiction
- $l - k = -1$ and $l + k = -1$, i.e. $l = -1$ and $k = 0$. Again, contradiction.

We can conclude that the proposition is true.

Exercise 5 *(10 points)*

Let n be a natural number and let $a_1, a_2, \ldots, a_n$ be a set of n real numbers. Prove that at least one of these numbers is less than, or equal to the average of these numbers. What kind of proof did you use?

We use a proof by contradiction.

Suppose none of the real numbers a_1, a_2, ..., a_n is less than or equal to the average of these numbers, denoted by $\overline{a}$.

By definition

$$\overline{a} = \frac{a_1 + a_2 + ... + a_n}{n}$$

Our hypothesis is that:

$$\begin{aligned} a_1 &> \overline{a} \\ a_2 &> \overline{a} \\ ... &> ... \\ a_n &> \overline{a} \end{aligned}$$

We sum up all these equations and get the following:

$$a_1 + a_2 + ... + a_n > n * \overline{a}$$

Replacing $\overline{a}$ in equation (9) by its value given in equation (4) we get:

$$a_1 + a_2 + ... + a_n \quad > \quad a_1 + a_2 + ... + a_n$$

This is not possible: a number cannot be strictly smaller than itself: we have reached a contradiction. Therefore our hypothesis was wrong, and the original statement was correct.

Exercise 6 *(5 points each; total 10 points)*

Let n be an integer. Show that if $n^3 + 9$ is even, the n is odd, using:

a) a proof by contraposition

b) a proof by contradiction

Let n be an integer. We define $P(n) : n^3 + 9$ is even, and $Q(n) : n$ is odd. We want to prove that $P(n) \rightarrow Q(n)$. We use two different proofs:

a) Proof by contrapositive

We want to show that $\neg Q(n) \rightarrow \neg P(n)$.

Hypothesis: $\neg Q(n)$, i.e. n is even. By definition of even numbers, there exists and integer k such that $n = 2k$. Then,

$$n^3 + 9 = 10k^3 + 9 = 2(5k^3 + 4) + 1$$

Since $5k^3 + 4$ is an integer, $n^3 + 9$ can be written in the form $2k' + 1$, where k' is an integer; therefore, $n^3 + 9$ is odd, i.e. $\neg P(n)$ is true.

a) Proof by contradiction

We suppose that $P(n) \rightarrow Q(n)$ is false, i.e. that $\neg(P(n) \rightarrow Q(n))$ is true, i.e. that $P(n) \wedge \neg Q(n)$ is true. This is only the case if $P(n)$ is true and $\neg Q(n)$ is true.

If $\neg Q(n)$ is true, then n is even. By definition of even numbers, there exists and integer k such that $n = 2k$. Then,

$$n^3 + 9 = 10k^3 + 9 = 2(5k^3 + 4) + 1$$

Since $5k^3 + 4$ is an integer, $n^3 + 9$ can be written in the form $2k' + 1$, where k' is an integer; therefore, $n^3 + 9$ is odd, i.e. $\neg P(n)$ is true. However, we have supposed that $P(n)$ is true: we have reached a contradiction. The original statement is therefore true.

Extra Credit *(5 points)*

Use Exercise 5 to show that if the first 12 strictly positive integers are placed around a circle, in any order, then there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 20.

Let a_1, a_2, ..., a_{12} be an arbitrary order of 12 positive integers from 1 to 12 being placed around a circle:

Since the twelve numbers a correspond to the first 12 positive integers, we get:

$$a_1 + a_2 + ... + a_{12} = 1 + 2 + ... + 12 = 78 \quad (1)$$

Notice that the a_1, a_2, ..., a_{12} are not necessarily in the order 1, 2, ..., 12. They do include however the twelve integers from 1 to 12: these is why the sum is 78

Let us now consider the different sums S_i of three consecutive sites around the circle. There are 12 such sums:

$$\begin{aligned}
S_1 &= a_1 + a_2 + a_3 \\
S_2 &= a_2 + a_3 + a_4 \\
S_3 &= a_3 + a_4 + a_5 \\
S_4 &= a_4 + a_5 + a_6 \\
S_5 &= a_5 + a_6 + a_7 \\
S_6 &= a_6 + a_7 + a_8 \\
S_7 &= a_7 + a_8 + a_9 \\
S_8 &= a_8 + a_9 + a_{10} \\
S_9 &= a_9 + a_{10} + a_{11} \\
S_{10} &= a_{10} + a_{11} + a_{12} \\
S_{11} &= a_{11} + a_{12} + a_1 \\
S_{12} &= a_{12} + a_1 + a_2
\end{aligned}$$

We do not know the values of the individual sums S_i; however, we can compute the sum of these numbers:

$$\begin{aligned}
S_1 + S_2 + ... + S_{12} &= (a_1 + a_2 + a_3) + (a_2 + a_3 + a_4) + ... + (a_{12} + a_1 + a_2) \\
&= 3 * (a_1 + a_2 + ... + a_{12}) \\
&= 3 * 78 \\
&= 234
\end{aligned}$$

The average of S_1, S_2, ..., S_{12} is therefore:

$$\begin{aligned}
\overline{S} &= \frac{S_1 + S_2 + ... + S_{12}}{12} \\
&= \frac{234}{12} \\
&= 19.5
\end{aligned}$$

Based on the conclusion of Exercise 5, at least one of S_1, S_2, ..., S_{12} is smaller to or equal to $\overline{S}$, i.e., 19.5. Because S_1, S_2, ..., S_{12} are all integers, they cannot be equal to 19.5. Thus, at least one of S_1, S_2, ..., S_{12} is smaller to or equal to 19.