Logic (2)

1. Conditional propositions

Consider the statement, "If you earn an A in logic, then I'll buy you a car". This is a compound proposition made of the two statements

p : "You earn an A in logic"

q : "I will buy you a car"

The original statement says "if p is true, then q is true", or, more simply, "if p, then q", or "p implies q". Using mathematical symbol, "$p \to q$".

Truth table for conditional

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

When p is false, $p \to q$ is true, no matter what the truth value of q is.

Think of $p \to q$ as a deal. If p is true, but q is false, we have broken the deal. If p is false, we do not break the deal. Note that $p \to q$ is not "p causes q".

__Property:__

$$(p \to q) \iff (\neg p) \vee q$$

__Proof:__

p	q	$p \to q$	$\neg p$	q	$\neg p \vee q$
T	T	T	F	T	T
T	F	F	F	F	F
F	T	T	T	T	T
F	F	T	T	F	T

The two propositions are equivalent as they share the same truth values

Notes:

$p \to \neg\neg p$ is a tautology

$p \to \neg p$ is <u>not</u> a contradiction!

Definitions :

(1) The statement $\neg q \to \neg p$ is called the contrapositive of the statement $p \to q$

Property :

A conditional and its contrapositive are equivalent

Proof :

$$p \to q \iff (\neg p) \vee q$$
$$\iff q \vee (\neg p)$$
$$\iff \neg(\neg q) \vee (\neg p)$$
$$\iff (\neg q) \to (\neg p)$$

(2) The statement $q \to p$ is called the converse of the statement $p \to q$

Note that a conditional and its converse are NOT equivalent.

Example : let p : "It rains"
q : "They cancel school"

$p \to q$: "If it rains, then they cancel school"

Converse: $q \to p$ "If they cancel school, then it rains"
(different meaning)

contrapositive $\neg q \to \neg p$: "If they do not cancel school, then it does not rain"
(similar meaning)

2. Biconditional

Definition: The biconditional, written $p \Leftrightarrow q$, is defined as the compound statement $(p \to q) \wedge (q \to p)$, for any propositions p and q.

Truth table

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Phrasing:

"p if and only if q"
"p is necessary and sufficient for q"
"p is equivalent to q"

3. Rules of inference

$p \rightarrow p \vee q$	Addition
$p \wedge q \rightarrow p$	Simplification
$(p) \wedge (q) \rightarrow p \wedge q$	Conjunction
$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus ponens
$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$	Modus tollens
$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Transitivity
$[(p \vee q) \wedge \neg p] \rightarrow q$	Syllogism
$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow q \vee r$	Resolution

Example 1

Let us consider the following assumptions:

(a) If it rains today, then we will not go on a canoe trip today

(b) If we do not go on a canoe trip today, then we will go on a canoe trip tomorrow

Can we conclude that:

If it rains today, then we will go on a canoe trip tomorrow

Proof:

let p: It rains today

q: we will not go on a canoe trip today

r: we will go on a canoe trip tomorrow

	Step	Reason
	$p \to q$	Hypothesis (a)
	$q \to r$	Hypothesis (b)
Therefore	$p \to r$	Transitivity

The rules of inference give us the tools to validate a conclusion from a set of hypotheses

Example 2

Let us consider a more complex set of assumption

a) " It is not sunny today and it is colder than yesterda

b) " If We will go swimming ~~only if~~ then it is sunny "

(c) " If we do not go swimming, we will have a barbe

(d) " If we will have a barbecue, then we will be hom by sunset"

Can we conclude, " We will be home before sunset"

Proof

let p : " It is sunny today"

q : " It is colder than yesterday "

r : " We will go swimming "

s : " We will have a barbecue "

t : " We will be home before sunse

	Step	Reason
1	$\neg p \wedge q$	Hypothesis (a)
2	$\neg p$	Simplification of step 1
3	$r \to p$	Hypothesis (b)
4	$\neg r$	Modus tollens using 2 and 3
5	$\neg r \to s$	Hypothesis (c)
6	s	Modus ponens using step 4 and 5
7	$s \to t$	Hypothesis (d)
8	t	Modus ponens using step 6 and 7

Methods of proof

Definition A theorem is a statement that can be shown to be true.

We demonstrate that a theorem is true with a sequence of statements that form an argument, called a proof.

To construct a proof, we need methods to derive new statements from old ones → this is where the rules of inference come in!

Possible statements in a proof:

Axioms, postulates
Proved theorems } underlying assumptions

Hypotheses / premises

Body of the proof } rules of inference

Conclusion

Example

Let p_1, p_2, and q be 3 propositions.

Show that:

$$[(p_1 \to q) \land (p_2 \to q)] \to ((p_1 \lor p_2) \to q)$$

Logical proof

	Step	Reason
1	$p_1 \to q$	hypothesis
2	$\neg p_1 \lor q$	property of implication, based on step 1
3	$p_2 \to q$	hypothesis
4	$\neg p_2 \lor q$	property of implication, based on step 3
5	$(\neg p_1 \lor q) \land (\neg p_2 \lor q)$	Conjunction (steps 2 and 4)
6	$(\neg p_1 \land \neg p_2) \lor q$	Distributivity
7	$[\neg (p_1 \lor p_2)] \lor q$	De Morgan's law
8	$(p_1 \lor p_2) \to q$	Property of implication, based on step 7

Direct proofs

If we want to prove $p \to q$, we show that if p is true, then q must also be true. This shows that the combination p True / q False cannot occur.

Example:

Theorem: Let n be an integer. Show that if n is even, then n^2 is even

Proof: Let n be an integer

Step	Reason
n is even	premise
There exists k such that $n = 2k$	Definition of even
$n^2 = 4k^2$	Definition of square
$n^2 = 2(2k^2)$	Factorization
n^2 is even	Definition of square.

Conclusion: The theorem is true for all n.

1.2) Indirect proofs

Since the implication $p \to q$ is equivalent to its contrapositive, $\neg q \to \neg p$, it can be proved by showing that its contrapositive is true. An argument of this type is called an indirect proof.

Example:

Let n be an integer.
Show that if n^3+5 is odd, then n is even

Proof.

We use the contrapositive: if n is odd then n^3+5 is even.
Let n be an integer.

	Step	Reason
	n is odd	premise
There exists k,	$n = 2k+1$	Definition of odd number
	$n^3 = (2k+1)^3$	Cube
	$n^3 = 8k^3+12k^2+6k+1$	Development
	$n^3+5 = 8k^3+12k^2+6k+6$	
	$n^3+5 = 2(4k^3+6k^2+3k+3)$	Factorization
	n^3+5 is even	Definition of even number
		Conclusion

Proof by contradiction

Suppose we can find a contradiction q such that $\neg p \rightarrow q$. Then $\neg p \rightarrow F$ is true, which is only possible if $\neg p$ is false. Consequently, p is true. This is a proof by contradiction.

Example: show that $\sqrt{2}$ is irrational

Proof: let p be $\sqrt{2}$ is irrational

	step	reason
1	$\neg p$ is true	hypothesis
2	$\sqrt{2}$ is rational	definition of $\neg p$
3	$\sqrt{2} = \frac{a}{b}$, $b \neq 0$, $\gcd(a,b) = 1$	Definition of rational number
4	$a = \sqrt{2}\, b$	Multiplication by b
5	$a^2 = 2b^2$	Square
6	a^2 is even	definition of even number
7	a is even	known result
8	$a = 2k$	definition of even
9	$4k^2 = 2b^2$	Replacement in step 5
10	$b^2 = 2k^2$	Division by 2
	b^2 is even	definition of even
	b is even	known result
	a and b even	Contradiction

iii) Proof by cases

To prove an implication of the form

$$(p_1 \vee p_2 \cdots \vee p_n) \longrightarrow q$$

We use the property:

$$(p_1 \vee p_2 \cdots \vee p_n) \to q \iff (p_1 \to q) \wedge (p_2 \to q) \wedge \cdots (p_n \to$$

Example

If n is not a multiple of 3, n^2 is not a multipl

Proof

n is not a multiple of 3 (Premise)

$n = 3k+1$ or $n = 3k+2$ (Definition of multiple of 3)

Case 1

$n = 3k+1$

$n^2 = 9k^2 + 6k + 1$

$n^2 = 3(3k^2 + 2k) + 1$

n^2 is not a multiple of 3

Case 2

$n = 3k+2$

$n^2 = 9k^2 + 6k + 4$

$n^2 = 3(3k^2 + 2k + 1) + 1$

n^2 is not a multiple of 3

Conclusion: n^2 is not a multiple of 3

Mistakes in proof

Problems with circular reasoning

$[p \wedge (p \to q)] \to q$: modus ponens

$[q \wedge (p \to q)] \to q$ circular reasoning !

The conclusion of the argument is used as one of the truth or principles upon which the argument rests.

Example :

Jones is an honest man
I know this is true, because he told me himself and he is an honest man !

Working backwards

Prove that $3^{1/3} > 2^{1/2}$

" Proof ":

$$3^{1/3} > 2^{1/2}$$
$$(3^{1/3})^6 > (2^{1/2})^6$$
$$3^2 > 2^3$$
$$9 > 8 \to \text{true} \ldots$$

Correct proof :

$9 > 8$ Ordering of natural number

$3^2 > 2^3$ Definition of power

$3^{2/6} > 2^{3/6}$ Raise to power $1/6$

$3^{1/3} > 2^{1/2}$ Conclusion

5. Predicates and quantifiers

5.1 Definitions

We have seen that "$x+3=1$" is not a proposition as we cannot assess its truth value.

We can denote the expression as $P(x)$, where x is the variable and P the <u>predicate</u>.

A predicate $P(x)$ can become a proposition through quantification.

There are two main types of quantification:

- <u>Universal quantification</u>

  $P(x)$ is true for all x in the universe Ω of discourse.

  <u>Notation</u>: $\forall x \in \Omega, \ P(x)$

  <u>Example</u>: $\forall n \in \mathbb{N}, \ n^2 \geq 0$

- <u>Existential quantification</u>

  There is a value x in the universe of discourse Ω such that $P(x)$ is true.

  <u>Notation</u>: $\exists x \in \Omega, \ P(x)$

  <u>Example</u> $\exists n \in \mathbb{N}, \ n$ is prime

tatement	When is it true?	When is it false?
$x \in \Omega, P(x)$	We show $P(x)$ true for all x	There exists an x for which $P(x)$ is false
$x \in \Omega, P(x)$	There is an x for which $P(x)$ is true	$P(x)$ is false for every x

.2 Negating quantifiers

Statement	Negation	Equivalent statement
$\forall x \in \Omega,\ P(x)$	$\neg(\forall x \in \Omega,\ P(x))$	$\exists x \in \Omega,\ \neg P(x)$
$\exists x \in \Omega,\ P(x)$	$\neg(\exists x \in \Omega,\ P(x))$	$\forall x \in \Omega,\ \neg P(x)$

.3 Theorems and quantifiers

.3.1 Existence proofs

Many theorems are assertions that object of a particular type exists:

$$\exists x \in \Omega,\ P(x)$$

There are two types of existence proofs:

a) <u>Constructive proofs</u> : find x explicitly

b) <u>Non-constructive proofs</u> : we do not find x such that $P(x)$ is true but we show that there must exist one.

<u>Examples</u>

<u>Constructive proof</u> Prove that there exists a pair of consecutive integers such that one of them is a perfect square, and the other is a perfect cube.

Perfect square: $\exists n \in \mathbb{N}, \quad a = n^2$

Perfect cube: $\exists p \in \mathbb{N}, \quad a+1 = p^3$ or $a-1 = p^3$

We observe that if $n = 3$ and $p = 2$, then $n^2 = 9$ and $p^3 = 8$. The pair of integers we are looking for is $(8, 9)$

Non - constructive proof.

Show that there escists a pair of irrational numbers a and b such that $c = a^b$ is rational.

Proof : We know that $\sqrt{2}$ is irrational.

Let us define $c = (\sqrt{2})^{\sqrt{2}}$

There are two cases:

- c is rational $\rightarrow$ we are done
- c is irrational

  Then let $d = c^{\sqrt{2}} = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \times \sqrt{2}} = \sqrt{2}^2 = 2$

  d is rational, and we are done.

We have proved that a and b esast, even though we do not know their values.

5.3.2 Uniqueness proofs

Sometimes theorems assert the existence of a unique element with a particular property P.

Proofs of such theorems require 2 steps:

→ Existence : find x such that $P(x)$ is true.

Uniqueness : show that if $y \neq x$, then $P(y)$ is false

. or

show that if $P(x)$ and $P(y)$ are true, then $x = y$

Example

theorem : let a be a real number $\neq 0$.

Let b and x be real numbers.

Show that there is a unique solution to the equation $0 = ax + b$.

Proof

Existence:

$$ax + b = 0$$

$$ax = -b$$

$$x = \frac{-b}{a} \quad (\text{as } a \neq 0)$$

Uniqueness. Let us assume that x_1 and x_2 are both solutions of the equation

$$ax + b = 0.$$

$$ax_1 + b = 0 \quad \text{and} \quad ax_2 + b = 0$$

$$ax_1 + b = ax_2 + b$$

$$ax_1 = ax_2$$

$$x_1 = x_2 \quad (\text{division by } a \neq 0)$$

5.3.3 Counter example

To show that a proposition of the form $[\forall x \in \Omega, P(x)]$ is false, it is enough to find one value of x such that $P(x)$ is false. That value is called a counter-example.

Example: Prove or disprove that

$$\forall n \in \mathbb{N}, \ 2^n + 1 \text{ is prime.}$$

Proof:

$n = 1 \qquad 2^n + 1 = 3$ prime

$n = 2 \qquad 2^2 + 1 = 5$ prime

$n = 3 \qquad 2^3 + 1 = 9$ not prime!

Therefore the proposition is not true.