Sequences and Summations

1) Sequences

Definition: A sequence is a function from a subset of the set of integers (usually either the set $\mathbb{Z}^+ = \{0, \dots\}$ or the set $\mathbb{N}$) to a set S. We use the notation a_n to denote the image of the integer n.

Example. Let $a_n = \frac{1}{n}$

$\rightarrow$ Sequence $1, \frac{1}{2}, \dots, \frac{1}{n}$

Note. By extension, the set $\{a_n\}$ is also called a sequence.

Definition A geometric progression is a sequence of the form:

$$a, ar, ar^2, \dots\dots ar^n$$

where the initial term a and the common factor r are real numbers.

Definition An arithmetic progression is a sequence of the form

$$a, a+r, \dots\dots a+nd$$

where the initial term a and the common difference d are real numbers.

2) Summations

Let us consider a sequence $a_n, a_{n+1}, \dots, a_m$.

The sum of all elements of this sequence is:

$$S = a_n + \dots\dots a_m$$

We use the notation:

$$S = \sum_{i=n}^{m} a_i$$

to represent the sum.

The variable i is called the index of summation, and the choice of the letter i is arbitrary.

$$S = \sum_{i=n}^{m} a_i = \sum_{j=n}^{m} a_j = \sum_{k=n}^{m} a_k$$

Summation are easy to implement in a program:

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S ← 0
For (i ← n ; i ≤ m ; Step = 1)
{
    S ← S + a_i
}
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Theorem: If a and r are real numbers and $r \neq 0$, then

$$\sum_{i=0}^{n} a r^i = \begin{cases} a \dfrac{r^{n+1} - 1}{r-1} & \text{if } r \neq 1 \\ (n+1) a & \text{if } r = 1 \end{cases}$$

Proof: Note first that $\sum_{i=0}^{n} a r^i = a \sum_{i=0}^{n} r^i$

Let us define $S_n = \sum_{i=0}^{n} r^i$

Then:
$$r S_n = \sum_{i=0}^{n} r^{i+1}$$

Let us define $j = i+1$. When $i=0$, $j=1$ and when $i = n$, $j = n+1$.

Therefore
$$r S_n = \sum_{j=1}^{n+1} r^j = \sum_{j=0}^{n} r^j + r^{n+1} - 1$$

Hence:
$$r S_n = S_n + r^{n+1} - 1$$
$$(r-1) S_n = r^{n+1} - 1$$

Two cases:

$r = 1$ $\qquad S_n = \sum_{i=0}^{n} 1 = n+1$ and $\sum_{i=0}^{n} a r^i = a(n+1)$

$r \neq 1$ $\qquad S_n = \dfrac{r^{n+1} - 1}{r-1}$ and $\sum_{i=0}^{n} a r^i = a \dfrac{r^{n+1} - 1}{r-1}$

Some useful summations

Sum	Closed form
$\sum_{i=0}^{n} a r^i, \ r \neq 0$	$a \dfrac{r^{n+1}-1}{r-1} \quad r \neq 1$ $a(n+1) \quad r = 1$
$\sum_{i=0}^{n} i$	$\dfrac{n(n+1)}{2}$
$\sum_{i=0}^{n} i^2$	$\dfrac{n(n+1)(2n+1)}{6}$
$\sum_{i=0}^{n} i^3$	$\dfrac{n^2(n+1)^2}{4}$

Some proofs:

Let us consider $S = \sum_{i=0}^{n} (i+1)^2$. We can compute S in 2 ways:

(i) $S = \sum_{j=1}^{n+1} j^2 = \sum_{j=0}^{n} j^2 + (n+1)^2$

(ii) $S = \sum_{i=0}^{n} (i^2+2i+1) = \sum_{i=0}^{n} i^2 + 2 \sum_{i=0}^{n} i + (n+1)$

Hence:

$$\cancel{\sum_{j=0}^{n} j^2} + (n+1)^2 = \cancel{\sum_{i=0}^{n} i^2} + 2\sum_{i=0}^{n} i + (n+1)$$

$$2\sum_{i=0}^{n} i = (n+1)^2 - (n+1) = n(n+1)$$

Therefore:

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

We start over with $S = \sum_{i=0}^{n} (i+1)^3$. It can be written in 2 ways:

(i) $S = \sum_{i=0}^{n} (i+1)^3 = \sum_{j=0}^{n} j^3 + (n+1)^3$

(ii) $S = \sum_{i=0}^{n} (i^3+3i^2+3i+1) = \sum_{i=0}^{n} i^3 + 3\sum_{i=0}^{n} i^2 + 3\sum_{i=0}^{n} i + (n+1)$

Therefore:

$$\cancel{\sum_{j=0}^{n} j^3} + (n+1)^3 = \cancel{\sum_{i=0}^{n} i^3} + 3\sum_{i=0}^{n} i^2 + 3\sum_{i=0}^{n} i + (n+1)$$

$$3\sum_{i=0}^{n} i^2 = (n+1)^3 - 3\sum_{i=0}^{n} i - (n+1)$$

$$3\sum_{i=0}^{n} i^2 = (n+1)^3 - 3\frac{n(n+1)}{2} - (n+1)$$

$$3\sum_{i=0}^{n} i^2 = (n+1)\left[\frac{2(n+1)^2 - 3n - 2}{2}\right]$$

$$= (n+1)\left[\frac{2n^2+4n+\cancel{2}-3n-\cancel{2}}{2}\right]$$

$$= \frac{(n+1)\,n\,(2n+1)}{2}$$

Hence

$$\sum_{i=0}^{n} i^2 = \frac{(n+1)\,n\,(2n+1)}{6}$$

Mathematical induction

Many theorems state that a proposition $P(n)$ is true for all (positive) integers n.

For example, $\forall n \in \mathbb{N}, \quad \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

We cannot prove such theorems by trying all possible values of n (there is an infinite number of them)!

Mathematical induction is a proof technique that usually allows to prove such assertion.

How does it work? Induction is equivalent to a "cascade" reaction. It works by first proving that the statement is true for a start value, and then by proving that the process used to go from one value to the next is valid. If both are true, then the statement is true for any value.

Think of it as a "domino effect":

If you have a long row of dominoes and if

(1) The first domino falls

(2) Whenever a domino falls, its next neighbor falls,

then all dominoes fall.

A proof by mathematical induction that a proposition $P(n)$ is true for every positive integer n consists of two steps:

<u>Basis step</u>: The proposition $P(i)$ is true, for a start position i (usually 0 or 1)

<u>Inductive step</u>: The implication $P(k) \rightarrow P(k+1)$ is shown to be true for every positive integer k.

The principle of mathematical induction allows then to conclude that $\forall n \in \mathbb{N}, n \geq i$, $P(n)$ is true.

Expressed as a rule of inference, it can be stated as:

$$[P(i) \wedge (\forall k \in \mathbb{N}, k \geq i, P(k) \rightarrow P(k+1))] \rightarrow \forall n \in \mathbb{N}, n \geq i \; P(n)$$

<u>Why mathematical induction is valid?</u>

We use a proof by contradiction:
We suppose we show that $P(i)$ is true, and that $\forall k \geq i$, $P(k) \rightarrow P(k+1)$. We also suppose that there is (at least) one value of $n \geq i$ such that $P(n)$ is not true. Let S be the set of values n for which $P(n)$ is not true. Then S is bounded from below (by i). Therefore S has a least element, N_S (well ordering property), with $N_S > i$. Then $N_S - 1$ does not belong to S, hence $P(N_S - 1)$ is true. But using the premise, since $P(N_S - 1)$ is true, $P(N_S)$ is true. This contradicts that $N_S \in S$. Hence mathematical induction is valid.

Examples:

(1) Use mathematical induction to prove that the sum of the first n odd positive integer is n^2, for all n.

Basis step: $P(1)$ states that the sum of the first odd integer is 1^2: this is true.

Inductive step: Let us suppose $P(k)$ is true, $k \geq 1$. That is, the sum of the first k odd integers is k^2:

$$S_k = 1 + 3 + 5 + \cdots\cdots 2k-1 = k^2$$

Then

$$S_{k+1} = S_k + 2k+1 = k^2 + 2k + 1 = (k+1)^2$$

This shows that $P(k+1)$ is true.
The principle of mathematical induction allows us to conclude that the proposition is true for all n.

(2) Use mathematical induction to prove that:

$$\forall n \in \mathbb{N}, \quad \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Basis step: $P(1)$: $\sum_{i=0}^{1} i^2 = 1$ and $\frac{1(1+1)(2+1)}{6} = 1$

Inductive step: Let us suppose $P(k)$ true, $k \geq 1$.

Then

$$S_{k+1} = \sum_{i=0}^{k+1} i^2 = S_k + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$S_{k+1} = \frac{(k+1)[2k^2 + k + 6k + 6]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

The principle of mathematical induction allows us to conclude that the proposition is true for all n.

Strong induction

There is another form of induction that is used to prove results: strong induction, or the second principle of mathematical induction.

Basis step: $P(i)$ is true

Inductive step: The implication

$$[P(i) \wedge P(i+1) \wedge \dots \wedge P(k)] \rightarrow P(k+1)$$

is shown to be true for every positive integer $k \geq i$.

The principle of strong induction allows then to conclude that $\forall n \geq i$, $P(n)$.

Example of application:

You have a chocolate bar, made of N small squares. How many cuts do you need to separate all squares?

Answer: $N-1$

Proof by strong induction:

Basis step:

$P(1)$ is true: You need 0 cuts to separate 1 square!

($P(2)$ is true: you need 1 cut to separate 2 squares)

Inductive step:

Let us suppose $P(i)$ is true, for all $i \leq k$.

Start with a chocolate bar with $k+1$ squares.

Break it into 2 bars, one with m_1 squares, the other with m_2 squares. Note that $m_1 + m_2 = k+1$.

To cut the chocolate bar with m_1 squares (m_2 squares) we need $m_1 - 1$ cuts ($m_2 - 1$ cuts).

The total number of cut is then: $T = m_1 - 1 + m_2 - 1 + \underbrace{1}_{\text{1st cut}} = k$

The principle of strong induction allows us to conclude that $P(n)$ is true for all n.

Example in geometry:

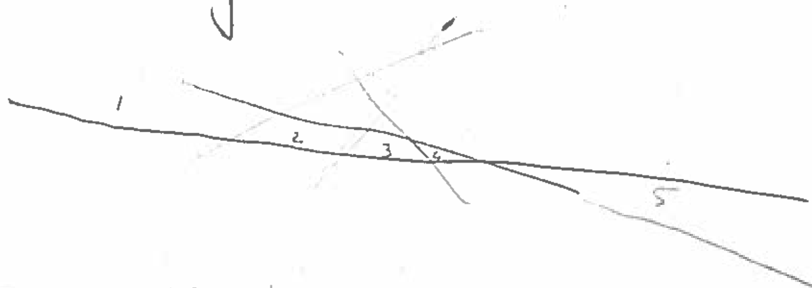
Show that n lines separate the plane into $\frac{n^2+n+2}{2}$ regions, if no two of these lines are parallel, and no three pass through a common point!

Proof: Basis step: One lines divides the plane in 2 regions and $\frac{n^2+n+2}{2} = \frac{1+1+2}{2} = 2$ when $n=1$

<u>Inductive step</u>: Suppose that $P(k)$ is true, $k \geq 1$.

The plane is cut into $\frac{k^2+k+2}{2}$ regions.

Let us add a $(k+1)$ line. It will cut all k lines, as we suppose that no 2 lines are parallel. To cut k lines, it will cross $(k+1)$ regions:



Each $(k+1)$ region is then divided into 2 regions by the line $(k+1)$. Therefore the line $(k+1)$ creates $(k+1)$ regions.

The total number of regions is:

$$T = \frac{k^2+k+2}{2} + k+1 = \frac{k^2+3k+4}{2} = \frac{(k+1)^2+(k+1)+2}{2}$$

The principle of mathematical induction allows us to conclude that the property is true for all n.

Recursive definitions

Sometimes it is difficult to define an object explicitly. It might be easier to define it with respect to itself $\rightarrow$ This is called a recursion.
We usually use 2 steps to define a recursive function:

Basis steps: specify the value of the function at 0 (and/or 1)

Recursive step: give a rule for finding its value at an integer n, from its value at smaller integer.

Example: The Fibonacci suite is defined as:

Basis step: $f_0 = 0$ and $f_1 = 1$

Recursive step: $f_n = f_{n-1} + f_{n-2}$

Applications:

Show that $\forall n \geqslant 3$, $f_n > \alpha^{n-2}$

with $\alpha = \frac{1+\sqrt{5}}{2}$ (golden number)

Proof:

Basis step: $n=3$ $f_3 = f_2 + f_1 = f_1 + f_0 + f_1 = 2$

and $\alpha^{n-2} = \alpha < 2$

Inductive step

Suppose that $P(i)$ is true, for all $3 \leq i \leq k$.
We want to show that $P(k+1)$ is true.

$$f_{k+1} = f_k + f_{k-1}$$

We know that $f_k > \alpha^{k-2}$ and $f_{k-1} > \alpha^{k-3}$

Therefore:

$$f_{k+1} > \alpha^{k-2} + \alpha^{k-3}$$

$$f_{k+1} > \alpha^{k-3}(\alpha + 1)$$

Note that $\alpha^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2} = 1+\alpha$

Then $$f_{k+1} > \alpha^{k-1}$$

The principle of strong induction allows us to conclude that the proposition is true for all n.

Application 2

Show that: $P(n)$: $f_1 + f_3 + \cdots f_{2n-1} = f_{2n}$.

Basis step: $P(1)$: $f_1 = 1$
$f_{2n} = f_2 = f_1 + f_0 = 1$ → Therefore $P(1)$ is true.

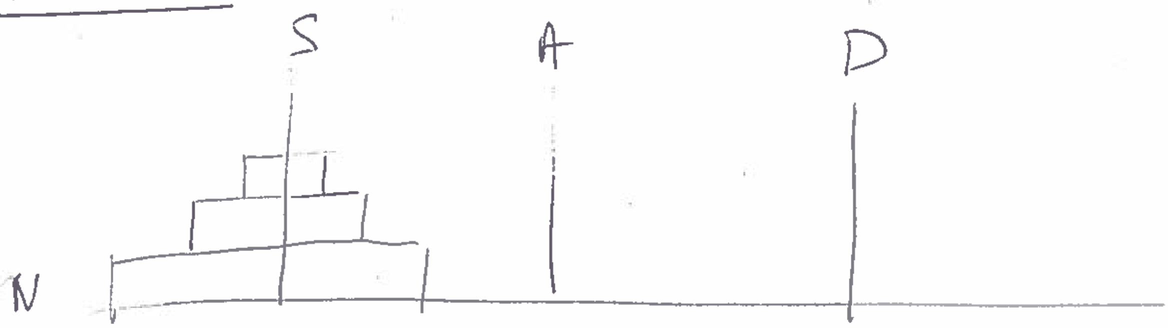
Inductive step: Suppose that $P(k)$ is true, $k \geq 1$

We compute: $S_{k+1} = f_1 + \cdots f_{2k+1} = f_1 + \cdots f_{2k-1} + f_{2k+1} = f_{2k} + f_{2k+1}$

Therefore $S_{k+1} = f_{2k+2}$, which validates $P(k+1)$

According to the principle of mathematical induction, we can conclude that $P(n)$ is true for all n.

Hanoi towers



Problem: Move N disks from the source (S) pin to the destination (D) pin, using an auxilary pin (A), such that you always have disks piled in decreasing order of size.

Let M_N be the optimal number of moves for solving the problem for N disks.

$N = 0 \qquad M_0 = 0$

$N = 1 \qquad M_1 = 1$

$N = 2 \qquad M_2 = 3$

For N disks: At some point during the procedure, you will have to move the largest disk, labeled N, from the source S to the destination D. To do that the (N-1) other disks must be on the auxiliary pin, A.
Let us suppose you moved these (N-1) disks from S to A optimally → you needed M_{N-1} moves.
After moving the disk N to D, you need to move the (N-1) others disks from (A) to D → you need M_{N-1} moves for this.

The optimal number of moves is therefore:

$$M_n = 2M_{n-1} + 1, \text{ with } M_0 = 0$$

To find a closed expression for M_n:

We write $u_n = M_n - b$

then

$$u_n + b = 2(u_{n-1} + b) + 1$$

$$u_n = 2u_{n-1} + b + 1 \rightarrow \text{we choose } b = -1$$

Therefore $u_n = M_n + 1$ or $M_n = u_n - 1$

Also: $u_n = 2u_{n-1}$, $u_0 = 1$

We show that $\forall n \in \mathbb{N}$, $u_n = 2^n$

Basis step $u_0 = 1$ and $2^0 = 1$

Induction step : Let us suppose $u_k = 2^k$, $k \geq 0$

Then $u_{k+1} = 2u_k = 2 \times 2^k = 2^{k+1}$

The principle of mathematical induction allows us to conclude that $u_n = 2^n \quad \forall n \in \mathbb{N}$

Therefore $\boxed{M_n = 2^n - 1 \quad \forall n \in \mathbb{N}}$