Analyzing and Characterizing Small-World Graphs *

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Abstract

We study variants of Kleinberg’s small-world model where we start with a $k$-dimensional grid and add a random directed edge from each node. The probability of a random edge is to $v$ is proportional to $d(u,v)^r$ where $d(u,v)$ is the lattice distance and $r$ is a parameter of the model.

For a $k$-D grid, we show that these graphs have poly-log expected diameter when $k < r < 2k$, but have polynomial expected diameter when $r > 2k$. This shows an interesting phase-transition between small-world and “large-world” graphs.

We also present a general framework to construct several classes of small-world graphs with expected diameter, which includes several existing different settings such as Kleinberg’s grid-based setting and tree-based setting [14].

We also generalize the idea of ‘probability $\propto$ the inverse distance’ to design small-world graphs. We use semi-metric and metric functions to abstract distance to create a class of random graphs where almost all pairs of nodes are connected by a path of length $O(\log n)$, and using only local information we can find paths of poly-log length.

1 Introduction

Small-world networks are an being used and studied in many disciplines, including the social and natural sciences. These networks possess a striking property, the so called small-world phenomenon, also often spoken of as “six degrees of separation” (between any two people in the United States)\(^\dagger\). Since many real networks exhibit small-world properties, a number of network models have been proposed as a framework to study this phenomenon. Watts and S. Strogatz [22] introduced a random graph setting to model certain small-world graphs. This model features two main properties, low average path length and significant clustering. We use small-world graphs to mean graphs with poly-log (expected) diameters, to focus on this property of small separation between nodes.

Recently, Kleinberg [15], building on the work of Watts and Strogatz [22], proposed a family of small-world networks to study another compelling aspect of Milgram’s findings: a greedy algorithm using only local information can construct short paths. Kleinberg adds a number of directed long-range random links to an undirected $n \times n$ lattice network. The long-range links have a non-uniform distribution which favors arcs to close nodes over more distant ones. These graph models have generated considerable interest and recent work. Applications have been found using Kleinberg’s small-world model or the ideas behind it to decentralized search protocols in peer-to-peer systems [20, 23], and gossip protocols for spreading information in a communication network [13].

Kleinberg’s model starts with a simple base graph and adds a number of randomly distributed arcs. The base graph models local “contacts”. The additional random links model potentially long-range contacts which can connect distant components and thus greatly shrink the diameter of the graph. Thus we see a promising formula: a simple base graph plus some random links can add nice properties (such as Kleinberg’s setting with expected small diameter and short greedy paths for all $s - t$ pairs). Kleinberg’s setting is a very

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\(^\dagger\)Milgram discovered this in his pioneering work in the 1960’s [21], and recent work by Dodds et al. suggests its still true [9].
specific one, so we ask: what are the essential features, underlying the distribution of random links and the grid structure which produce these nice properties? We address this question in this paper in two ways. First, we mostly complete the picture of the diameter problem in Kleinberg’s grid-based setting by identifying the critical point where the graph changes from expected poly-log to expected polynomial diameter, depending on how much we favor links to close nodes. Then we construct a framework, which starts with a very abstract model (an arbitrary base graph and some general rules for adding random arcs), then refine our model to identify properties which lead to small expected diameter and later to allow us to find short paths using local information only.

Some of our graphs have small expected diameter, yet need not use a distance measure to describe the random link distribution\(^2\). Kleinberg’s models (grid-based setting [15], tree-based and group-induced settings [14]) and several other well-known small-world graphs fit our abstract models and thus can be analyzed using our general results on diameter and routing. Moreover, we introduce or generalize several techniques used for bounding a graph’s diameter.

We briefly review Kleinberg’s setting then summarize our results in the next subsection. Kleinberg’s basic model uses a two-dimensional grid as a base with long-range random links added between any two nodes \(u\) and \(v\) with a probability proportional to \(d^{-2}(u,v)\), the inverse square of the lattice distance between \(u\) and \(v\). In the basic model, each node has an undirected local link to each of its four grid neighbors and one directed long-range random link. A straightforward extension of this basic model is to have multiple random links from each node and use a \(k\)-dimensional grid for any \(k = 1, 2, 3, \ldots\); also use an inverse \(r^{th}\) power distribution of the random links, for any real constant \(r\), instead.

In [19], we proved a tight \(\Theta(\log n)\) bound for the expected diameter of Kleinberg’s extended model: for a \(k\)-dimensional grid and an inverse \(r^{th}\) power distribution when \(0 \leq r \leq k\), i.e. for \(0 \leq r \leq 2\) in the 2-D case or for \(0 \leq r \leq 1\) in the 1-D case. However, the diameter problem for \(r > k\) was open before this paper. Note that the complexity of greedy routing in Kleinberg’s grid-based setting has already been analyzed. For \(r = k\) it takes \(\Theta(\log^2 n)\) expected steps while for \(r \neq k\), greedy routing takes expected polynomial time\([15, 2, 19, 11]\).

### 1.1 Our results

First, we mostly complete the analysis of the diameter problem in Kleinberg’s grid-based setting. For a \(k\)-D grid, we show that the model still has poly-log expected diameter when \(k < r < 2k\), but has polynomial expected diameter when \(r > 2k\). However, interestingly enough, the case \(r = 2k\) is still open, though our initial experiments suggest that the model is a large-world. In particular, for Kleinberg’s 1-D model, for any \(r < 2\) the expected diameter is upper-bounded by poly-log functions \(O(\log n)\) for \(r \leq 1\), however, for \(r > 2\), the expected diameter can be lower bounded by a (low-degree) polynomial function. This creates an interesting phase-transition between small-world and “large-world” graphs.

We also present a framework to construct several classes of small-world graphs with \(\Theta(\log n)\) expected diameter, which includes several existing different settings such as Kleinberg’s grid-based setting and tree-based setting [14]. We start this framework with a very abstract class of random graphs, then we gradually add in conditions to achieve more refined classes, which are more likely small-world candidates.

We also employ the idea of ‘probability ∝ the inverse distance’ to design small-world graphs. Again, we start with a general class, based on an abstract semi-metric function (abstracted from the use of distance), and then add in refining criteria to construct a hierarchy of classes with interesting properties. As a result, we obtain an abstract class of random graphs such that under some easy conditions, for almost all pairs of nodes, we can find an ideal path of length \(O(\log n)\), and using only local information we can find paths of expected poly-log length.

### 1.2 Related work

There has been considerable work on the small-world phenomenon. See [16] for early surveys and [15] for a more recent account on modeling small-world networks. Before Kleinberg’s model, Watts and Strogatz [22] proposed a refined model by randomly rewiring the edges of a ring lattice each with a probability parameter

\(^2\)Thus, links no longer favor close nodes over distant nodes.
p. Watts and Strogatz observed that for small $p$ the model reflects many practical small-world networks with small typical path length and a non-negligible clustering coefficient. Kleinberg has generalized his basic model in several ways in [14] including a generalization that encompasses both lattice-based and tree-based ("taxonomic" or "hierarchical") small-world networks.

The diameter of random graphs is a classic problem [5, 6, 7, 10] but most results use uniformly distributed arcs. Bollobas and Chung [6], study a graph model very similar to Watts and Strogatz in [22] with the nodes of a cycle (or a "ring") randomly matched to form additional long-range links. The closest diameter work with non-uniform arc probabilities is on long-range percolation graphs (LRPGs) which have been used to study physical properties. As in Kleinberg’s model, a grid with (undirected) local links is augmented by undirected long-range random links whose probability is inversely related to their distance. Note that in contrast to Kleinberg’s model, the added links are undirected, and the degree of a node is not fixed. Thus the analysis techniques here are somewhat different than those to analyze Kleinberg’s and related models. Benjamini and Berger study the diameter of 1-D LRPGs [3] and Coppersmith et al. extend this to $k$-D grids [8]. Both papers prove diameter results which show how the expected diameter changes as the arc probability parameters change. Biskup improves these results by proving tighter bounds [4]. These papers show there are critical points where the expected diameter changes from constant, to poly-log and then to polynomial as the probability parameter changes. We show some similar transitions occur in Kleinberg’s setting.

There have also been several recent papers which analyze greedy routing in other small-world like networks [1, 2, 14, 17, 19, 11]. Though our focus is on diameter results, we show how to incorporate greedy-like routing (to find short paths) into an abstract class which already has expected $O(\log n)$ diameter.

2 Basic concepts

To generalize Kleinberg’s small-world models, we develop an abstract class of random graphs, which includes Kleinberg’s small-world settings (in [15, 14]). We then use this abstract class as a platform to create a general framework to analyze the diameter (and other related issues) in a variety of settings.

Consider the following random assignment (or matching) operation: for a given node $u$ in a graph $G$, make a random trial under a specific distribution rule $\tau$ to find another node $v$. We write this as $v \xleftarrow{R_{\tau}} u$ or $v = R_{\tau}(u)$. For example, in Kleinberg’s basic grid setting, $\tau$ is defined as having $v \xleftarrow{R_{\tau}} u$ with probability proportional to the inverse square of the lattice distance between $u$ and $v$, i.e. $Pr[v \xleftarrow{R_{\tau}} u] \propto d^{-2}(u,v)$. We can think of a random graph constructor using this operation which forms a family of random graphs. We use a given base graph $H$ and a compatible graph constructor, where each additional $(u,v)$ link (with $v \xleftarrow{R_{\tau}} u$) is called a random link. Random links are generated for a node, not for pairs of nodes as in traditional random graphs\footnote{Even when we use undirected random links, we can consider that: each node $u$ generates and, so, owns certain random links, while some other random links also incident to $u$ are not owned by $u$ but by some other nodes (which generated these links)}. This operation is implicitly used in Kleinberg’s small-world models [15, 14].

We restrict the distribution rules ($\tau$) we use to ones which have the following property: each $R_{\tau}$ call performs an independent trial. Multiple $R_{\tau}$ calls on the same input node ($u$), also are independent trials. We now define an abstract class of random graphs, which includes all of Kleinberg’s small-world settings.

**Definition 1.** Given a set of undirected base graphs $\mathcal{H}$, a distribution $\tau$ and a constant integer $q \geq 1$, a family of random graphs $\mathcal{F}_R G(\mathcal{H}, \tau, q)$ consists of graphs, each of which is a base graph $H \in \mathcal{H}$ plus at least $q$ out-going random links\footnote{They are directed by our default assumption.} generated under distribution $\tau$ for each node.

All the families of random graphs we consider in this paper are $\mathcal{F}_R G$ families. For example, Kleinberg’s basic grid model ([15]) is a $\mathcal{F}_R G(\mathcal{H}, \tau, q)$ family, where $\mathcal{H}$ consists of all $n \times n$ grids ($n = 1, 2, 3, \ldots$), $q = 1$, and $\tau$ is the inverse square distribution. Note that there is no restriction on the set of fixed edges $E$ in the base graphs. For example, the fixed edges can be the local links in Kleinberg’s grid model, a complete graph, or nothing at all as in Kleinberg’s tree-based model.

We present new diameter results for Kleinberg’s grid settings, which complement previous diameter results. In §3 we start with the most basic setting, i.e. the (one-dimensional) cycle augmented by random
links as follows. Define \( C(r, n) \) as the setting where nodes are labeled \( 0, 1, 2, \ldots, n-1 \) and each node \( i \) has 2 undirected local links to \( (i-1) \mod n \) and \( (i+1) \mod n \) for \( 0 \leq i \leq n-1 \), and has one directed random link to some node \( j \). The probability its random link is to \( j \), is proportional to \( |i-j|^r \), where \( r \geq 0 \) is a parameter to be specified.

We then generalize our approach in §4 (for analyzing Kleinberg’s grid model) and introduce several abstract families of random graphs which can be constructors for small-worlds. From these abstract families, by adding some proper additional conditions, we obtain different classes of small-world graphs with poly-log expected diameter. In §5 we create classes with short paths which can be found by decentralized algorithms (using local information only), and present a generalization of §3’s results.

We now consider some useful basic lemmas. Consider a family \( \mathcal{F} = \mathcal{FRG}(\mathcal{H}, \tau, q) \) and a graph \( G \in \mathcal{F} \), which has base graph \( H = (V, E) \).

**Lemma 1.** For any graph \( G \) from a family \( \mathcal{FRG}(\mathcal{H}, \tau, q) \), any two disjoint subset of vertices \( S \) and \( T \) chosen without any knowledge of the random links from \( S \), the probability of having a random link from some node in \( S \) to at least one node in \( T \), is \( \Pr[S \to T] \geq 1 - e^{-\eta |T||S|} \) where \( \eta = \epsilon(S, T) \) denotes the minimum value of the probabilities of all the random links between \( S \) and \( T \), i.e. the minimum value of \( \Pr[R_{\tau}(u) = v] \) for all \( u \in S \) and \( v \in T \).

**Proof.** Given an arbitrary node \( u \in S \), let \( p \) denote \( \Pr[u \text{ misses } T] \), i.e. none of the \( q \) random links from \( u \) goes to any node in \( T \), and similarly, let \( P = \Pr[S \text{ misses } T] \). A given random link from \( u \) goes to \( T \) with probability at least \( \epsilon |T| \), therefore it is easy to see that, \( p \leq (1 - \epsilon |T|)^n \). Using the basic calculus fact \( 1 + x \leq e^x \), we have \( p \leq e^{-\eta |T|} \). Now combining all the events \( u \text{ misses } T \) for each \( u \in S \), we have \( P \leq e^{-\eta |T||S|} \). Therefore, \( \Pr[S \to T] = 1 - P \geq 1 - e^{-\eta |T||S|} \).

We use lemma 1 several times later, where we often have the sizes of \( S \) and \( T \) large enough so that \( \epsilon |T||S| = \Omega(\log n) \) and thus, for some \( \theta > 0 \), \( \Pr[S \to T] \geq 1 - O(n^{-\theta}) \), which tends to 1 when \( n \) goes to the infinity. So, (we say) almost surely, \( T \) is apart from \( S \) by just one random link.

**Lemma 2.** If each of \( n \) events \( \{B_i\}_{i=1}^n \) occurs with probability at least \( 1 - p \), where \( p < 1/n \), then the combining event \( \bigcap_{i=1}^n B_i \) occurs with probability at least \( 1 - np \).

**Proof.** Using the Union Bound law, we have \( \Pr[\bigcup_{i=1}^n \overline{B_i}] \leq \sum_{i=1}^n \Pr[\overline{B_i}] \leq np \), hence \( \Pr[\bigcap_{i=1}^n B_i] = 1 - \Pr[\bigcup_{i=1}^n \overline{B_i}] \geq 1 - np. \)

Note that lemma 2 applies even if the \( B_i \) are not independent.

### 3 Diameter transitions in Kleinberg’s model

For simplicity, we first look at the basic model \( C(r, n) \) and then extend our results to more general settings in §3.3. For \( 0 \leq r \leq 1 \) this cycle setting is known to have expected \( \theta(\log n) \) diameter [19]. We now consider the diameter of \( C(r, n) \) when \( r > 1 \).

#### 3.1 The \( C(r, n) \) setting with \( 1 < r < 2 \)

We first present our notations and basic definitions, then a sketch of our basic approach, and finally our theorems and proofs in detail.

For \( r > 1 \), the normalized coefficient \( L = \frac{1}{\sum_{d=r}^{\infty} d^{-r}} = O(1) \); in fact, \( L^{-1} \leq 2 \sum_{d=1}^{\infty} d^{-r} \), which converges to a constant, depending on \( r \) only\(^5\). So, \( \Pr[i \to j] = L |i-j|^{-r} = \theta(|i-j|^{-r}) \). Let \( I_l(u) \) or \( I_l^u \) denote a ‘segment’ of length \( l \), starting at node \( u \), i.e. \( I_l^u = \{u, (u+1) \mod n, \ldots, (u+l-1) \mod n\} \).

Consider segment \( I_x^u \) of length \( x \) for some arbitrary node \( u \). Let \( 0 < \xi < 1 \). Divide \( I_x^u \) into \( x^{1-\xi} \) (disjoint) subsegments of length \( x^{\xi} \). Let \( \mathcal{D}_x = \{J_1, J_2, \ldots, J_{x^{1-\xi}}\} \) be this set of subsegments, i.e. \( J_k = I_x^u(u+(k-1)x^{\xi}) \) for \( 1 \leq k \leq x^{1-\xi} \). For simplicity, we assume \( x, x^{1-\xi} \) and the like are integers.

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\(^5\)This bound can be approximated by \( \int_1^\infty x^{-r} = \frac{1}{1-r} x^{1-r} \bigg|_1^\infty = \frac{1}{1-r} \).
Definition 2. For each node $u$, $I^u_x$ is $\xi$-complete if for any ordered pair of segments $(J_i, J_k)$ from $D_\xi(I^u_x)$, there is an edge crossing from $J_i$ to $J_k$.  

Let $\delta(I^u_x)$ be the diameter of the subgraph induced by nodes in the segment $I^u_x$. Here, $\delta(I^u_x)$ is a random variable and has a determined value for each instance of our random graph (once the random links are determined). $E[\delta(I^u_x)]$ is independent of position $u$ and we let $\delta_x = E[\delta(I^u_x)]$.

The main idea. In order to upper bound the diameter of our random graph in this 1-D setting, we use a probabilistic recurrence approach. We establish a (probabilistic) relation between the diameter of a segment and that of a smaller one. In particular, we relate $\delta(I_x)$ (the diameter of a segment of length $x$) to $\delta(I_y)$, where $y = x^\xi$ for some $\xi \in (0,1)$. With high probability, $\delta(I_x)$ is bounded by a constant multiple of $\delta(I_y)$. Thus, we use standard recurrence techniques to bound $\delta_n$ (the graph’s diameter) based on $\delta_x$, for a small initial length $x_0$ (so $\delta_x$ is upper bounded by a poly-log function of $n$).

We use this crucial observation: $I_x$ is almost surely $\xi$-complete for $x$ and $\xi < 1$ large enough. So, $\delta(I_x)$ is almost surely not larger than twice the maximum diameter of any subsegment in $D_\xi(I_x)$. We formalize the above ideas in the following lemmas and then prove our main theorem.

Lemma 3. For $r/2 < \xi < 1$, there exists a constant $c > 0$ so that, for $x \geq c \ln \frac{1}{\xi - \frac{r}{2}} n$,

$$Pr[I^u_x \text{ is } \xi\text{-complete}, \forall u = 0..n-1] \geq 1 - n^{-2}$$

Note that for $0 < \xi < 0.5$, $I^u_x$ is not $\xi$-complete for any $u$. Since $x^\xi$, the number of random links from nodes in a subsegment $J_i \in D_\xi(I_x)$, is smaller than $x^{1-\xi-1}$, the number of other subsegments $J_k \in D_\xi(I_x)$.

Proof. We need to lower bound the probability of the event that there exists an edge connecting $J_a$ and $J_b$ for all possible pairs $(J_a, J_b)$. Using lemma 1, $Pr[J_a \rightarrow J_b] \geq 1 - e^{\varphi |J_a||J_b|}$, where $\epsilon = \epsilon(J_a, J_b)$. Note that $|J_a| = |J_b| = x^\xi$ while $\epsilon(J_a, J_b) = Lx^{-r}$, so

$$Pr[J_a \rightarrow J_b] \geq 1 - e^{\varphi |J_a||J_b|} = 1 - e^{-Lx^{-r} \times x^{2\xi}} = 1 - e^{-L(-x^{2\xi - r})}$$

(1)

$I_x$ is $\xi$-complete if there exists an arc between $J_a$ and $J_b$ for all possible pairs $(J_a, J_b)$. The number of such pairs is $< x^{2(\xi)}$, hence using lemma 2, $P_x = Pr[I_x \text{ is } \xi\text{-complete}] \geq 1 - (e^{-Lx^{2\xi - r}} \times x^{2\xi})$. Let $E$ be the event that $I^u_x$ is $\xi$-complete, $\forall u = 0..n-1$. Again, using lemma 2:

$$Pr[E] \geq 1 - n(1 - P_x) \geq 1 - (n e^{-Lx^{2\xi - r}} \times x^{2\xi})$$

Now, we choose $c = (5/L)^{\frac{1}{2\xi - r}}$, and hence, for $x \geq x_0 = c \ln \frac{1}{\xi - \frac{r}{2}} n$,

$$Pr[E] \geq 1 - (n e^{-5\ln n \times x^{2\xi}}) \geq 1 - (n^{-1} \times x^{2\xi}) \geq 1 - n^{-2}$$

since $x^{2\xi} < n^2$.

The next two results follow directly.

Lemma 4. If a segment $I^u_x$ is $\xi$-complete then $\delta(I^u_x) \leq 2 \max_{J \in D_\xi(I_x)} \delta(J) + 1$.

Corollary 1. If $I^u_x$ is $\xi$-complete for each $u = 0..n-1$ then $\max_{u = 0..n-1} \delta(I^u_x) \leq 2 \max_{u = 0..n-1} \delta(I^u_x) + 1$.

Theorem 1. For any $r$ such that $1 < r < 2$, there exists a constant $\beta$ such that the expected diameter of $C(r, n)$ is $O(\log^\beta n)$.

Proof. Since $r < 2$ we can choose $r/2 < \xi < 1$. Let $\phi(x)$ be a random variable s.t. $\phi(x) = \max_{u = 0..n-1} \delta(I^u_x)$. $\phi(x)$ is determined for each instance of our random graph and is clearly non-decreasing in $x$. If $I^u_x$ is $\xi$-complete for all $u = 0..n-1$ then from corollary 1, $\phi(x) \leq 2\phi(x^\xi) + 1$. Thus from lemma 3, there exists a constant $c > 0$ so that, for $x \geq x_0 = c \log \frac{1}{\xi - \frac{r}{2}} n$,

$$Pr[\phi(x) \leq 2\phi(x^\xi) + 1] \geq 1 - n^{-2}$$

(2)

If we think of a super-graph with the $J_i$’s as it’s nodes then these crossing links make it a complete graph.

Although our approach is similar to Karp’s [12], his theorems necessity conditions are not met here.
We can use a standard recurrence technique to upper bound \( \phi(n) \), based on \( \phi(x_0) \) and \( n \) only.

Define the sequence \( \{x_i\}_{i=0}^{t+1} \), where \( x_{i+1} = x_i^b \) with \( b = 1/\xi \), \( x_0 = c \log^{1/\xi} n \), and

\[
t = \left\lfloor \log_b (\log_{x_0} n) \right\rfloor = \left\lfloor \frac{\log \log n}{\log b} \right\rfloor = \frac{\log \log n}{\log b} + O(1)
\]

Thus, \( x_t \leq n < x_{t+1} \). Now we look closer at this sequence \( \{\phi(x_i)\}_{i=0}^t \) and use (2) to upper bound the last term (which differs from \( \phi(n) \) by a multiple constant, based on the first term and \( t \). We claim that each of the events \( E_t : \phi(x_i) \leq 2\phi(x_{i+1}) + 1^n, i = 1, 2, \ldots, t \) and \( E_{t+1} : \phi(n) \leq 2\phi(x_1) + 1^n \) occurs with probability at least \( 1 - n^{-2} \). The first \( t \) events can be justified directly from (2), while we can also easily extend our proof of lemma 4 to justify the last event. Let \( E \) be the event that \( E_1, E_2, \ldots, E_{t+1} \) all occur. Using lemma 2, \( E \) occurs with probability at least \( 1 - (t+1) \times n^{-2} \leq 1 - O(n^{-1}) \).

It is easy to see that event \( E \) implies \( \phi(x_i) \leq 2^t \phi(x_0) + 2^{t+1} - 1 \), \( \forall i = 1, t \) and thus, \( \phi(n) \leq 2^{t+1} \phi(x_0) + 2^{t+1} - 1 \leq O((\log n)^{\log_b 2}) \times \phi(x_0) \). That is \( \Pr[\delta(I_n) \leq O(\log \beta n)] \geq 1 - O(n^{-1}) \) where \( \beta = \log_{1/\xi} 2 + \frac{1}{\xi^{1/\xi}} \). Note that \( \phi(x_0) \leq x_0 = O(\log^{2/\xi} n) \). Thus, \( \Pr[\delta(I_n) \leq O(\log \beta n)] \) tends to 1 when \( n \) goes to infinity, and almost surely \( \delta(I_n) = O(\log \beta n) \).

Note that our bound on \( \beta \) grows rapidly as \( r \) approaches 2.

### 3.2 The \( \mathcal{C}(r,n) \) setting with \( 2 < r \)

**Theorem 2.** For \( r > 2 \), \( \mathcal{C}(r,n) \) is a ‘large’ world with expected diameter \( \Omega(n^{\frac{2}{r-2} - o(1)}) \).

**Proof.** Let \( \frac{1}{r-1} > \gamma < 1 \). For any node \( i \), the probability that \( i \)'s random contact is at most a distance \( n^\gamma \) from \( i \), is \( 1 - O(\frac{n^{d_m n^\gamma}}{d^r}) = 1 - O(n^{-\gamma(r-1)}) \). Using lemma 2, the probability that all random links have length at most \( n^\gamma \), is \( \geq 1 - n \times O(n^{-\gamma(r-1)}) = 1 - O(n^{1-\gamma(r-1)}) \). Since \( \frac{1}{r-1} < \gamma \), this probability tends to 1 when \( n \) goes to infinity. Thus the diameter is at least \( \frac{n}{n^\gamma} = n^{1-\gamma} \) with overwhelming probability (tending to 1 when \( n \) goes to infinity). So, the expected diameter is \( \Omega(n^{\frac{2}{r-2} - o(1)}) \)

### 3.3 Extended settings

It is easy to extend our results for the 1-D settings without wraparound. The normalized coefficient for random links from a node \( i \) depends on the position of \( i \), i.e. \( \Pr[i \to j] = L(i) \times |i - j|^{-r} \). Now, \( \frac{n}{d=1} d^{-r} \leq L^{-1}(i) \leq 2 \frac{n^{1/2}}{d=1} d^{-r} \), i.e. \( L \leq L(i) \leq 2L \), so, equation (1), and hence the rest of our arguments, still apply.

We now consider the general \( k \)-D setting for \( k = 1, 2, 3 \ldots \). Let \( H(k, r, n) \) denote a \( k \)-dimensional hypercube \( H_n \) (an \( n \times n \times \ldots \times n \) hypercube) with undirected edges between adjacent nodes and one random directed link from each node where \( \Pr[u \to v] \propto d^{-r} \). The model is still a small-world when \( r < 2k \) but a ‘large-world’ when \( r > 2k \).

**Theorem 3.** For each \( k, r \) with \( k < r < 2k \), there exists \( \beta > 0 \) s.t. \( H(k, r, n) \) has expected diameter \( O(\log \beta n) \). For each \( k, r \) with \( 2k < r \) there exists \( \alpha > 0 \) s.t. \( H(k, r, n) \) has expected diameter \( \Omega(\log^\alpha n) \).

**Proof (sketch).** It is not hard to see that the same approach (and techniques) as before still apply, but we need to modify some details. We focus on \( k < r < 2k \) (we omit \( 2k < r \) which is simpler).

To establish a (probabilistic) recurrence relation, we now use \( k \)-D hypercubes (in place of segments in the 1-D setting). Consider a hypercube \( H_x \) of size \( x \) (in each dimension). As before, for some \( 0 < \xi < 1 \), we can divide \( H_x \) into \( x^{k(1-\xi)} \) disjoint hypercubes each of size \( x^{\xi} \) (in each dimension). Let \( D_{\xi}(H_x) = \{J_1, J_2, \ldots, J_x^{k(1-\xi)} \} \) denote this set of sub-hypercubes. If \( H_x \) is \( \xi \)-complete, i.e. there is a crossing edge from \( J_i \) to \( J_k \) for any pair \( (J_i, J_k) \), then as before, we have \( \delta(H_x) \leq 2 \max \delta(J) + 1, J \in D_{\xi}(H_x) \). So, we have the desired recurrence relation and can go on as before to justify that \( \delta(H_n) \) is almost surely upper bounded by a poly-log function. Therefore, the remaining concern is on the \( \xi \)-completeness of any \( H_x \) (for \( x \) large enough); more specifically, we need the following fact, a new version of lemma 3: there exists \( \xi \in (0, 1) \) and \( x_0 = O(\log^{\beta} n) \) for some \( \beta > 0 \) such that for \( x > x_0 \), any \( H_x \) is almost surely \( \xi \)-complete for \( n \) large enough. Again, we need to consider \( \Pr[J_a \to J_b] \) for any \( J_a, J_b \in D_{\xi}(H_x) \). Using lemma 1, \( \Pr[J_a \to J_b] \geq 1 - e^{-\epsilon |J_a||J_b|} \), where \( \epsilon = \epsilon(J_a, J_b) \). Note that \( |J_a| = |J_b| = x^{k\xi} \) while \( \epsilon(J_a, J_b) \leq \theta(x^{-r}) \).
So, $Pr[J_a \rightarrow J_b] \geq 1 - e^{-\theta(\epsilon - r) \times 2^{\frac{2k}{\epsilon}} = 1 - e^{\theta(\epsilon - 2k\epsilon - r)}}$. Now, by choosing any $\xi \in (\frac{1}{2\epsilon}, 1)$, we can go on as before (with lemma 3) to finish proving this fact.

Note that the case $r = 2k$ is open, however initial experiments (for the 1-D setting only) suggest that the setting has polynomial expected diameter. We generalize further the case $k < 2r < 2k$ in §5.2 (where we abstract distance to create an abstract class induced by semi-metric or metric functions).

4 Constructing $O(\log n)$ diameter graphs with non-uniform random links

To analyze the shortest path between a source node $s$ and a destination node $t$, we construct two subset chains which can be viewed as two trees with root from $s$ and $t$ and then show they intersect. Each subset in $s$’s subset chain contains nodes which can be reached directly from the preceding subset, and hence, can be reached from $s$. The subset chain from $t$ has a similar structure but contains nodes with links towards $t$. To show that the shortest $s - t$ path has length $O(\log n)$, the main idea is to show that each subset chain grows exponentially in size before they intersect\(^8\).

Exponential growth will be likely if each time we grow a new subset, with high probability more than one link from each node leaves the current subset. This was true in Kleinberg’s grid setting [19] (we called this: “link into or out of a ball” property). We now include this feature to refine our basic class $\mathcal{F}(H, \tau, \epsilon)$ and set $\mathcal{H} \subseteq \mathcal{H}$ plus at least $q$ out-going random links generated under distribution $\tau$ for each node. A random link $(u, v)$ is generated when we have $v \approx u$ (or $v = R_r(u)$).

**Definition 3.** For constants $\mu > 0$ and $\xi > 0$, family $\mathcal{F} = \mathcal{F}(H, \tau, \epsilon)$ meets ‘the $(\mu, \xi)$ expansion criterion’, or $\mathcal{F}$ is $(\mu, \xi)$-EXP, if for every $H = (V, E) \in \mathcal{H}$, for any node $u \in V$ and any arbitrary subset $C \subseteq V$ with size $\leq n^\mu$, for $v \approx u$, the probability that $v \notin C$ is at least $\xi$:

$$\forall u \in V, \forall C \subseteq V \text{ s.t. } |C| < n^\mu : Pr[v \approx u : v \notin C] \geq \xi \quad (3)$$

For example, from [18], it is easy to verify that Kleinberg’s grid setting with wrap-around distance $(\mu, 1 - o(1))$-EXP for any fixed positive constant $\mu < 1$. This criterion supports diversity and fairness in the distribution of random links: For a random link from any node, no small set of vertices (with size $\leq n^\mu$) can take most of the chance to have this link come into it (i.e., Don’t give too much to a small group).

**Definition 4 (Type $\mu$-Expansion).** For a constant $\mu > 0$, type $\mu$-Expansion contains all the families $\mathcal{F}(H, \tau, \epsilon)$ which meet $(\mu, \xi)$-EXP for some $\xi > 1/q$.

We define a node operation $\chi$, called an ‘expansion function’, as follows. Given any $u \in V$, this operation will call operation $R_r$ $q$ times. Also, let $\chi(u)$ denote the set of vertices from these $q$ $R_r$ calls. Thus the random links for graph $G$ are formed by performing operation $\chi$ on each node. From an arbitrary set $S$, we can construct a new set $\chi(S)$ by taking the union of all the $\chi(u)$’s for all $u \in S$: $\chi(S) = \bigcup_{u \in S} \chi(u)$.

Consider a family $\mathcal{F}$ of type $\mu$-Expansion, where $\mu > 0$. Let $\beta = q\xi^2$ (so $\beta > 1$). For any node $u$ and set $C$ of size less than $n^\mu - q$, which is determined before $\chi(u)$ is known, the expected number of fresh elements generated by $\chi(u)$ that do not belong to $C$ is greater than $\beta$: $E[|\chi(u) - C|] > \beta > 1$. Since $\chi(u)$ ‘contributes’ more than one expected fresh element outside of $C$, $\chi$ can be used to generate a chain of subsets from a small initial subset such that with high probability, the subsets will quickly grow to size $\Theta(n^\mu)$.

4.1 The out-going subset chain

Let $\mathcal{F}$ be a $\mu$-Expansion family for some fixed $\mu > 0$ and $G = (V, E)$ be an arbitrary graph from $\mathcal{F}$. Now, from an arbitrary initial set $S_0 \subseteq V$, we construct a chain of subsets $\{S_k\}$, namely the out-going subset chain with respect to the initial set $S_0$, s.t. $S_{k+1} = \chi(S_k) - \cup_{i=0}^k S_i; k = 1, 2, 3, \ldots$ Thus, $S_i$ is the nodes at distance $i$ from $S_0$ using random links. The following results show the subset chain will grow rapidly if the initial set $S_0$ is large enough.

\(^8\)Alternatively, each subset chain grows exponentially to a threshold, so they intersect with high probability.
Lemma 5. Let $\alpha = \alpha(n) = \theta(n^{\mu})$. For any sets $C, S \subset V$ s.t. $S \subset C$, $|C| \leq \alpha$, and $|S| = \Omega(\log n)$, almost surely $|\chi(S) - C|/|S| > \gamma$ for a constant $\gamma > 1$. Thus, by choosing $|S|$ large enough

$$\exists \gamma > 1, \forall \theta > 0, \exists \epsilon > 0 : |S| > c \log n \Rightarrow \Pr[|\chi(S) - C|/|S| > \gamma] = 1 - O(n^{-\theta})$$

The above lemma (proof in appendix) provides a probabilistic lower bound $\gamma$ on the growth rate of the subset chain in each early step (by choosing $C = \bigcup_{i=0}^{k} S_i$ to apply the lemma in each step). This growth rate can be maintained for a while as long as the subset sizes are still under a threshold. Let $\alpha = \alpha(n) = \theta(n^{\mu})$ and once we find the first $S_i$ with size $\geq \alpha$, we set all the next subsets equal to $S_i$.

For any $S \in V$ with size $\Omega(\log n)$, the subset chain originated from $S_0 = S$ will almost surely grow exponentially in size before reaching $\alpha$. Also, by choosing a sufficiently large constant $c$ s.t. $|S| > c \log n$ we will almost surely have $|S_i| > \alpha$ for some $k = O(\log n)$, i.e. $\exists k = O(\log n) : \Pr[|S_k| \geq \alpha] = 1 - O(n^{-\theta})$ for some $\theta > 0$. Moreover, this can be true for any given $\theta > 0$ by choosing $c$ large enough.

4.2 The in-coming subset chain

We now construct a subset chain, based on the random links coming to the nodes into the sets of the chain. We use a new ‘expansion function’ $\psi$, which is a counterpart of $\chi$, so we can reuse the formalism used in \S4.1 on the out-going subset chain and obtain similar results. Function $\psi$ is not state-less as $\chi$ was. For any subset of vertices $D$ and a node $u \in V$ we define $\psi(u, D)$ to return the set of all nodes $v \notin V$ s.t. $v$ has a random link to $u$. As before, $\psi(T, D) = \bigcup_{u \in T} \psi(u, D)$ for any subset $T$. Now, from an arbitrary subset $T_0 \subset V$, we can construct a chain of subsets $\{T_k\}$, namely the in-coming subset chain with respect to the initial set $T_0$, s.t. $T_{k+1} = \psi(T_k, D)$ for $k = 1, 2, 3, \ldots$, where $D = \bigcup_{i=0}^{k} T_k$. Similar to definition 3, we have:

**Definition 5.** For constants $\mu > 0$ and $\xi > 0$, family $\mathcal{F}$ meets the $(\mu, \xi)$ incoming expansion criterion, or $\mathcal{F}$ is $(\mu, \xi)$-IE, if the following is satisfied. For any subset $D$ with size at most $n^{\mu}$, for any node $u \in D$, if we generate $R_r(u)$ once for each node $v \notin D$ then the probability that $\exists v : R_r(v) = u$ is at least $\xi$:

$$\forall D : |D| < n^{\mu}, \forall u \in D : \Pr\left[\exists v \notin D : R_r(v) = u\right] > \xi \quad (4)$$

Similarly as with $\mu$-Expansion, for a fixed $\mu > 0$, we define type $\mu$-$\text{IncExp}$, which includes all the $\mathcal{F} \mathcal{R} \mathcal{G}(\mathcal{H}, \tau, q)$ families which meet $(\mu, \xi)$-IE where $\xi > 1/q$. For a $\mu$-$\text{IncExp}$ family, lemma 5 holds if we replace the use of function $\chi$ by that of function $\psi$ and subset $C$ by subset $D$ \(^{10}\). There is an interesting implication between these two expansion criteria for a large class of families. We call a family of random graphs, using a distribution $\tau$, $\delta$-symmetric (or just symmetric if $\delta = 1$) for some constant $\delta \geq 1$, if $\frac{\Pr[R_r(v) = u]}{\Pr[R_r(u) = v]} \leq \delta$ for all pairs of nodes $(u, v)$. It is easy to see that Kleinberg’s grid settings (using the inverse power distributions) have this property, and they are symmetric if wrap-around distance is used.

**Lemma 6.** If family $\mathcal{F}$ is $(\mu, \xi)$-EXP, for $0 < \mu, \xi < 1$, and is $\delta$-symmetric for some $\delta \geq 1$ then $\mathcal{F}$ is $(\mu, 1 - e^{-\xi/\delta})$-IE.

**Proof (sketch).** We need to prove (4). Let $p(u, v) = \Pr[R_r(u) = v]$ and $F$ be the event that $\exists v \notin D : R_r(v) = u$. The lemma is shown as $\Pr[F] = \prod_{v \notin D} (1 - p(v, u)) \leq \prod_{v \notin D} e^{-p(v, u)} = \exp\{-\frac{1}{\delta} \sum_{v \notin D} p(u, v)\} \leq e^{-\frac{1}{\delta}}$. Note that $\sum_{v \notin D} p(u, v) = \Pr[\exists v \notin D : R_r(u) = v] \geq \xi$. □

4.3 Abstract classes of small-world graphs

We refine the above families by adding conditions to obtain small-world graphs. If our graph is from a family in both types $\mu_1\text{-Expansion}$ and $\mu_2\text{-IncExp}$ for some $0 < \mu_1, \mu_2 < 1$ then, given any source $s$ and destination $t$, we can use the following strategy to construct a $\log n$-length path from $s$ to $t$.

First, we want to find subsets $S_0$ and $T_0$ of $\Omega(\log n)$ size using (local) edges in the base graph $H$ such that $s$ can reach any node in $S_0$ using $O(\log n)$ edges, and $t$ from $T_0$ by $O(\log n)$ edges. We then construct the out-going subset chain from $S_0$ and the in-coming subset chain from $T_0$. Our above results show that, with

\(^{9}\)In our real construction, we stop after finding this size.

\(^{10}\)The constructions of both subset chains share the same formalism.
overwhelming probability, there exists subset $S_k$ with size $\theta(n^{\mu_1})$ and $T_l$ with size $\theta(n^{\mu_2})$ s.t. any nodes in $S_k$ can be reached from $S_0$ by $O(\log n)$ links, and $T_0$ from $T_l$ by $O(\log n)$ links also. We now consider proper conditions so we can easily reach $T_l$ from $S_k$.

A simple solution is to consider the minimum probability $\epsilon = \epsilon(\tau)$, i.e. the minimum value of $Pr[R_r(u) = v]$ for all $u \neq v$. If $\epsilon(\tau)$ is large enough, then $T_l$ is almost surely apart from $S_k$ by at most one link.

**Definition 6 (Expansion Family).** A $\mathcal{FRG}(\mathcal{H}, \tau, q)$ is an Expansion family if it is $(\mu_1, \xi)$-EXP and $(\mu_2, \xi)$-IE for some constants $\xi > 1/q, \mu_1, \mu_2 > 0$, and $\epsilon(\tau) = \Omega(n^{-\mu_3})$ for a constant $\mu_3 < \mu_1 + \mu_2$.

We now justifiy that if our graph is an Expansion family then almost surely $T_l$ is apart from $S_k$ by at most one link (or the trees already intersected). We can assume all the nodes in $S_k$ are fresh (we have not known their random links yet$^{(11)}$) and hence, using lemma 1, $Pr[S_k \rightarrow T_l] \geq 1 - e^{-\epsilon[T_l]} \geq 1 - e^{-\Omega(n^{\mu_1+\mu_2-\mu_3})} \geq 1 - O(n^{-1})$, which tends to 1 when $n$ goes to the infinity.

The graphs from an Expansion family$^{(12)}$ are small-worlds, i.e. their expected diameter is poly-log in $n$, as long as each node is rich enough in neighbors$^{(13)}$ such that large enough initial subsets (i.e. $S_0, T_0$) can be formed. Without this final condition, however, there is not even a guarantee that these graphs are connected. If there are no edges in the base graph ($E = \emptyset$) then even with the added random edges, the graphs can be unconnected; an example of that will be presented in the next subsection.

We now add in new notions for neighboring in the base graphs. A node $u$ is called $k$-neighbored for some $k \in \mathbb{N}$ if there exist at least $k$ nodes and each of them is connected to $u$ by a path of length $\leq k$ (in the undirected base graph). For example, if $u$ belongs to a connected component of size $k$ in the base graph, then $u$ is $k$-neighbored. A base graph $H = (V, E)$ is called $k$-neighbored if all the nodes are $k$-neighbored. A connected graph is $k$-neighbored for all $k \leq |V| - 1$. Now, for $k$ large enough, $k$-neighbored graphs allows us to construct large enough initial subsets. The next theorem now follows fairly directly$^{(14)}$.

**Theorem 4.** For any two nodes $s, t$ in a graph of an Expansion family, if $s$ and $t$ are $c \log n$-neighbored for some constant $c > 0$ then there almost surely exist $O(\log n)$-length paths between $s$ and $t$. An Expansion family, using $(c \log n)$-neighbored base graphs where $c > \frac{\delta \xi}{(q^{\xi}-1)}$, has expected diameter $O(\log n)$.

Therefore, for a graph from an Expansion family, if we remove all small connected (in the base graph) components of size $\leq c \log n$, we obtain a small-world with expected diameter $O(\log n)$.

**Using super-nodes.** We now consider random graphs which use $c \log n$-neighbored base graphs.

**Theorem 5.** Consider a family $\mathcal{FRG}(\mathcal{H}, \tau, q)$, which is $(\mu_1, \xi)$-EXP and $(\mu_2, \xi)$-IE for some constants $\xi, \mu_1, \mu_2 > 0$, where $\epsilon(\tau) = \Omega(n^{-\mu_3})$ for some constant $\mu_3 < \mu_1 + \mu_2$, and $\mathcal{H}$ contains $c \log n$-neighbored base graphs. Then there almost surely exists a path of length $O(\log n)$ between any two nodes.

**Proof.** This theorem is a simple corollary of the previous theorem if $q$ is s.t. $\xi > 1/q$. However, for $q < Q = \left[\frac{1}{\xi}\right]$ the theorem still holds. The main idea is to form super-nodes with $Q$ random links. The $c \log n$-neighbored property assures that we can always partition the graph into super-nodes each of which is a subgraph of constant diameter and has at least $Q$ random links. The length of a path constructed here differs by only a constant from before (when we have $q \geq Q$).

These abstract classes for (almost) small-world graphs are broad enough to accommodate many different well-known small-world models: Bollobas and Chung’s [6], Watts and Strogatz’s [22], Kleinberg’s grid-based [15], tree-based, and group-induced models [14]. Kleinberg describes his group-induced model with well-known small-world models: Bollobas and Chung’s [6], Watts and Strogatz’s [22], Kleinberg’s grid-based [15]. Kleinberg describes his group-induced model with two abstract properties, and it is not hard to see that the second property implies our $(\mu, \xi)$-IE for some $0 < \mu, \xi < 1$. We show that our results apply to Kleinberg’s tree-based model in the following section. It is relatively straight-forward to extend this case for similar results in the group-induced model.

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$^{(11)}$ We omit a conditioning issue: if we construct $s'$ subset chain (s-SSC) first then the growth of $t'$ subset chain (t-SSC) is conditioned on the existence of s-SSC and vice versa. Thus, we need to include $\cup_{l=1}^L S_l$ to $\mathcal{D}$ $(\text{§4.2})$ or $\cup_{l=2}^L T_l$ to $\mathcal{C}$ (4.1). Therefore, if $\mu_1 > \mu_2$, then we construct t-SSC first, otherwise s-SSC first.

$^{(12)}$ Note that we can construct similar classes by using $\mu$-Expansion and $\delta$-symmetric property instead.

$^{(13)}$ Informally, for any node in a expansion graph, either it is in a small isolated component or ‘the world is small’ for it, i.e. the graph except these small isolated pieces is a small-world.

$^{(14)}$ In fact, a full proof of it is very similar to that of theorem 14 in our previous work [19].
4.4 The diameter of a tree-based random graph

We now use our framework to analyze the diameter of Kleinberg’s tree-based (or “hierarchical network”) model [14] and its variants. Kleinberg shows that decentralized routing can be applied in more settings (not only the grid-based [15]), but even when no lattice structure appears at all (say, the network of the Web’s hyper-links). Kleinberg also introduces a group-induced model, a generalization of both grid-based and tree-based models [14]. He shows that using these models, greedy routing takes expected time $O(\log n)$ if nodes have out-degree $\theta(\log^2 n)$, and $O(\log^4 n)$ if the degrees are bounded by a constant (the constant-degree model). Note that for the constant-degree model, there is a fair chance that some nodes have no in-coming link at all. Thus, the routing protocol is only required to find a path from a source $s$ to a small neighborhood of a destination $t$ (a “cluster” in [14]), say, a small ‘subtree’ which contains the leaf $t$.

We now show our results with respect to the diameter of the tree-based model (which can be extended to the group-induced model for similar results). Basically, we show that when the degree of each node is at least 3, the setting is an Expansion family, that is by adding sufficient local links to make each node rich enough in neighbors, the graph will have diameter $O(\log n)$. In Kleinberg’s tree-based model, nodes are the leaves of a complete (for simplicity) $b$-ary tree $T$, where $b$ is a constant. Let $h(u,v)$ denote the height of the least common ancestor of $u$ and $v$ in $T$. There are no local links in this setting but there are a number of directed random links leaving each node $u$, under a distribution $\tau$, where a link is to $v$ with probability proportional to $h^{-h(u,v)}$.

Suppose there are exactly $q$ directed random links leaving each node. It is easy to see that the graphs in this tree-based setting are very likely unconnected (similar to the case of lacking local links in the grid-based setting [18]), however, the setting can still be an Expansion family by adding proper conditions. From [14], the normalizing coefficient of this link distribution is $\theta(\log^{-1} n)$. So, $\epsilon(\tau) = \theta(n^{-1}\log^{-1} n)$; thus, to have an Expansion family we need this setting to meet $(\mu_1 \xi)$-EXP and $\mu_2 \xi)$-IE for some $\xi > 1/q$ and $\mu_1 + \mu_2 > 1$. Consider the following fact which holds even if $q = 1$.

**Fact 7.** For Kleinberg’s tree graphs with $q = 1$, given a positive $\theta < 1$, a node $u$ and $C \subset V$ with size at most $n^{\theta}$, the probability that a random link from $u$ hits a node outside of $C$ is more than $1 - \theta - o(1)$ when $n$ is large enough. Also, the probability that there is a random link to $u$ from outside of $C$ is more than $1 - o(1)$ (i.e. almost $1 - e^{-1}$) when $n$ is large enough.

We omit the proof of this fact, from which it is easy to see that the setting meets $(x - o(1),1 - x)$-EXP and $(y - o(1),1 - e^{-1})$-IE for any $0 < x,y < 1$. Therefore, for a given value of $q$, we need to find $x,y$ s.t. $x + y > 1$; $q(1 - x) > 1$; $q(1 - e^{-1}) > 1$.

Solving this system of equations, we find $q \geq 3$.

**Theorem 6.** For each value of $q \geq 3$, the family of random graphs in Kleinberg’s tree-based setting is an Expansion family.

We can add in local links to make the base graph connected or make the base graph $c \log n$-neighbored: ring all the nodes in the base graph $H$ or alternately, ring all the subtrees of height at most $\log_0(c \log n)$. With $c$ determined as in theorem 4, this setting will have expected diameter $O(\log n)$.

5 Random graphs induced by semi-metric or metric functions

So far we have abstracted away topological features of Kleinberg’s grid setting with our expansion criteria, which creates our classes with the strongest having $O(\log n)$ expected diameter. We now generalize the use of a distance measure, especially in the distribution of random links (this makes greedy routing work). We design classes of random graphs using distributions based on semi-metric functions: we define a semi-metric function $d(u,v)$ and generate random links between any two nodes $u$ and $v$ with probability $\propto d^{-\tau}(u,v)$.

Consider a pair $(G,d)$ of a graph $G = (V,E)$ and a function $d = d_G : V^2 \to \mathbb{R}^+$ associated with $G$. We define $d$ to be a semi-metric function if for any $u,v \in V$, $d(u,v) = 0 \iff u = v$ and $d(u,v) = d(v,u)$. We define $N_k(u) = \{ v \in V | d(u,v) \leq k \}$, the subset of nodes within ‘distance’ $k$ from $u$. For $c_1, c_2 > 0$, graph $G$ is called $(c_1, c_2)$ linear-expanded with respect to $d$ if $\forall u \in V, k \in \mathbb{N} : c_1 \leq \frac{N_k(u)}{k} \leq c_2$ if $N_{k-1}(u) \neq V$, i.e. $|N_k(u)|$ grows nearly-proportionally to $k$ before $N_k(u)$ becomes $V$. 
Definition 7 (**InvDist family**). An *InvDist*(r) family is a \( \mathcal{F}(H, \tau, q) \) family where there exists constants \( c_1, c_2 \geq 0 \) s.t. for each base graph \( H \in \mathcal{H} \) there is an associated metric-function \( d \) s.t. \( H \) is \((c_1, c_2)\) linear-expanded w.r.t. \( d \), and where \( \Pr[R_r(u) = v] \propto d^{-\tau}(u, v) \).

All Kleinberg's small-world models (grid-based, tree-based and group-induced) fall into *InvDist*(1) for an appropriate kind of \( d \). For example, for Kleinberg's two-dimensional grid model [15], we can define \( d(u, v) \) as the square of the lattice distance between \( u \) and \( v \); for Kleinberg's group-induced model [14], we can define \( d(u, v) \) as the size of the minimum set containing both \( u \) and \( v \).

**Theorem 7.** If any \( \delta > 0, c_2 > c_1 > 0 \), there exists a constant \( q \geq 1 \) s.t. any \( \delta \)-symmetric *InvDist*(1) family specified by \( c_1, c_2 \) and \( q \) (as in definition 7) is an Expansion family.

For any graph from a *InvDist*(1) \( \delta \)-symmetric family using \( c \log n \)-neighbored base graphs for some constant \( c > 0 \), there almost surely exists an \( O(\log n) \) length path between any two nodes [17].

**Theorem 8.** If any \( \delta > 0, c_2 > c_1 > 0 \), there exists a constant \( q \geq 1 \) such that any \( \delta \)-symmetric *InvDist*(1) family specified by \( c_1, c_2 \) and \( q \) (as in definition 7) is an expansion family.

Note that if we only use undirected random links then the condition of \( \delta \)-symmetry is not necessary.

**Proof.** Consider a *InvDist*(1) family and a graph from it. We show that we can choose \( q \) big enough such that this family is an expansion family, i.e., it meets \((\mu_1, \xi_1)\)-FE and \((\mu_2, \xi_2)\)-SE for some \( \xi_1 > 1/q \) and \( \mu_1, \mu_2 > 0 \), and \( \frac{1}{c_1(\tau)} = o(n^{\mu_1 + \mu_2}) \) (where \( \epsilon(\tau) \) is the minimum probability given to a random link by \( \tau \)).

In order to justify the first expansion property, we need to estimate this type of probability: \( \Pr[v \sim u : a \leq d(u, v) \leq b] \) for \( 1 \leq a < b \leq K_u \), where \( K_u = \min \{ k \in \mathbb{N} | N_k(u) \neq \emptyset \} \), the maximum distance of any other node from \( u \). Note that, this probability is related to \( h(a, b) = \sum_{b=1}^{\infty} \frac{\ln b}{b^\alpha} \approx \ln(b/a) \) as will be shown later. Also, from \( c_1 < \frac{|N_k|}{k} < c_2 \), clearly, \( K_u = \theta(\ln r) \).

Let \( A(a, b) = \sum_{k=a}^{b} \left( |N_k(a)| - |N_{k-1}(u)| \right) \times \frac{1}{k} \) where \( |N_k(u)| - |N_{k-1}(u)| \) is the number of nodes at distance \( k \) from \( u \). Thus, \( 1/A(1, K_u) \) is the normalized coefficient \( C_u \) (of the distribution of the random links at \( u \)). So, \( \Pr[v \sim u : a \leq d(u, v) \leq b] = A(a, b)/A(1, K_u) \). Now, we have

\[
A(a, b) = \sum_{k=a}^{b} \left( |N_k(u)| - |N_{k-1}(u)| \right) \times \frac{1}{k} = \sum_{k=a}^{b-1} \frac{|N_k(u)|}{k} \quad \text{as } k \to \infty
\]

Now note that \( c_1 < \frac{|N_k|}{k} < c_2, i.e., \sum_{k=a}^{b-1} \frac{|N_k(u)|}{k+1} \leq c_1 \frac{c_2}{c_1} \), so

\[
c_1 h(a + 1, b) + c_1 - c_2 \leq A(a, b) \leq c_2 h(a + 1, b) + c_2 - c_1 \quad \text{OR } A(a, b) = \theta(\ln(b/a)) \quad (5)
\]

Now we justify the first expansion property. For any \( C \in V \) with size \( O(n^\mu) \) and \( 0 < \mu < 1 \), we consider \( \Pr[R_r(u) \notin C] \). Define \( k \in \mathbb{N} \) such that \( |N_k(u)| \leq |C| = O(n^\mu) < |N_0(u)| \). Thus, \( |N_k(u)| = O(n^\mu) \) and \( k = O(n^\mu) \). Again, using the observation that a ball is the best shape for \( C \) to minimize the probability \( \Pr[R_r(u) \notin C] \), it is easy to see that

\[
\Pr[R_r(u) \notin C] \geq \Pr[R_r(u) \notin N_k(u)] = \Pr[v \sim u : k + 1 \leq d(u, v) \leq K_u] = A(k + 1, K_u)/A(1, K_u).
\]

\( \xi \)From (5), \( A(k, K_u) \geq c_1 h(k, 2, K_u) + c_1 - c_2 \geq c_1 \ln(n^{\mu_1 + \mu_2}) + O(1) = c_2(1 - \mu) \ln n + O(1). \)

Also, \( A(1, K_u) \leq c_2 \ln n + O(1) \). Thus, \( \Pr[R_r(u) \notin C] \geq \frac{c_1(1 - \mu)}{c_2}, i.e., the family meets \((\mu, \xi_1)\)-FE for any \( 0 < \mu < 1 \) and \( \xi_1 = \frac{c_2(1 - \mu)}{c_2} \). Using \( \delta \)-symmetry, the family meets \((\mu, \xi_2)\)-SE where \( \xi_2 = 1 - e^{-\delta_1 \beta} > 0 \). Thus we need to choose \( \beta > \xi_2 \).

Now we only need to assure that \( \frac{1}{c_1(\tau)} = o(n^{\mu_1 + \mu_2}) \), i.e., \( \frac{1}{c_1(\tau)} = o(n^{2\mu}) \). Note that \( c_1(\tau) = \frac{1}{K_u A(1, K_u)} = O(\frac{1}{n \ln n}). \) Thus we only need to choose \( \mu > 0.5 \) then this condition is met.

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15 Note that any family satisfying all these criteria except having \( c_1 \leq \frac{|N_k|}{k} \leq c_2 \) instead (for some given constant \( \beta > 0 \)) can be normalized by using function \( d^s(u, v) = d^s(u, v) \) instead, and hence becomes an *InvDist*(r) family where \( r' = r/\beta \).

16 It is not hard to see that the second property (of the two abstract properties Kleinberg uses to describe his group-induced model) implies that \( |N_k(u)| \) grows nearly-proportionally to \( k \).

17 Note that if we only use undirected random links then the condition of \( \delta \)-symmetry is not necessary. In fact, theorem 8 is also true for *InvDist*(r), where \( 0 < r < 1, \) however the proof of this is fairly tedious.
5.1 A class with expected $O(\log n)$ diameter and $O(\log^2 n)$ length greedy-like paths

This section constructs a new class where the graphs have expected diameter $O(\log n)$ and greedy-like paths\(^{18}\) with expected length $O(\log^2 n)$. It is possible to use only local information to find good short paths of expected length $O(\log^2 n)$, and to design (off-line) algorithms to find ideal $O(\log n)$ paths.

We restrict $d(u,v)$ to be a ‘light’ metric by adding the condition that $d(u,v) \leq \alpha (d(u,w)+d(w,v))$ for any nodes $u,v,w$ and for some constant $\alpha$ (which is less strict than the triangle inequality). We define class $\mathcal{METR}(r)$ as class $\text{InvDist}(r)$ but induced by this type of light metric function instead. All Kleinberg’s small-world models (grid-based, tree-based and group-induced) are $\mathcal{METR}(1)$ families with the function $d(u,v)$ derived naturally from each model’s context. Except for the 1-D and the tree-based setting, this function is not a metric. For example, for the tree-based setting, we can define $d(u,v)$ as the size of the smallest subtree containing both $u$ and $v$, which is a metric. For the abstract group-induced model, we use $d(u,v)$ as the size of the smallest group containing both nodes $u$ and $v$, which generally does not satisfy the triangle inequality (but satisfies ours for an appropriate $\alpha$).

We now add appropriate neighboring conditions so routing algorithms using only local information can be used. For any $u \in V$, and integer $k$, let $B_k(u)$ denote the sub-graph induced by the set of nodes $v$ such that $d(u,v) \leq k$. An undirected base graph $H(V,E)$ is called $k$-strongly neighbored for an integer $k$ if for each $u \in V$ $B_k(u)$ is connected.

**Theorem 9.** For any graph from a $\mathcal{METR}(1)$ family using $c \log^2 n$-strongly neighbored base graphs for some constant $c > 0$, there exists a greedy-like path of expected length $O(\log^2 n)$ between any two nodes. Also, there almost surely exists a greedy-like path of length $O(\log^2 n)$ between any two nodes.

**Proof (Sketch).** We just apply Kleinberg’s idea of using phases to route from a source $s$ to a destination $t$, each phase aims at halving the remaining distance. Let $l = c \log^2 n$; assume $q = 1$ (worst case) and suppose $d(u,t) = 2k$ for the current node $u$. We consider how many links it takes to go from $u$ to any node $v \in N_k(t)$ in a greedy-like fashion: from $u$, we use ‘local links’ (of the base graph) to traverse $B_l(u)$ (assuming we know the local topology of the base graph) until we find a node $w$ which has a random link to a node $v \in N_k(t)$, then take this random link to $v$. Assume $2k > l$ (otherwise we can use local links to go directly to $t$). Consider $Pr[R_v(w) \in N_k(t)]$ where $w \in N_l(u)$. For any node $v \in N_k(t)$, note that

$$d(w,v) \leq \alpha (d(w,u) + d(u,v)) \leq \alpha l + \alpha^2 (d(u,t) + d(t,v)) \leq 2k \alpha + 3 \alpha^2 = k(2 \alpha + 3 \alpha^2)$$

Thus, $Pr[R_v(u) \in N_k(t)] \geq C_u \frac{|N_k(t)|}{k(2 \alpha + 3 \alpha^2)} = \theta(\log^{-1} n)$ since $C_u = \theta(\log^{-1} n)$ and $c_1 \leq |N_k(t)|/k \leq c_2$. Since $|N_l(u)| = |N_{c \log^2 n}(u)| = \theta(\log^2 n)$, we almost surely find such node $w \in N_l(u)$ with a random link to $N_k(t)$; thus it almost surely takes $\theta(\log^2 n)$ links to halve the remaining distance. Moreover, just like Kleinberg’s proof of his $O(\log^2 n)$ upper bound in [15], we can argue that each phase takes expected $O(\log n)$ links and then the obtained greedy-like path has expected length $O(\log^2 n)$ (after $\log n$ phases).

Now, combining theorems 8 and 9, we have the following result.

**Theorem 10.** For any graph from a $\mathcal{METR}(1)$ $\delta$-symmetric family using $c \log^2 n$-strongly neighbored base graphs for some constant $c > 0$, there almost surely exists a path of length $O(\log n)$ and a greedy-like path of expected length $O(\log^2 n)$ between any two nodes.

It is not hard to see that this theorem applies for Kleinberg’s grid model, and also for the tree-based and group-induced with proper local links (to make the base graphs become $c \log^2 n$-strongly neighbored).

5.2 $\mathcal{METR}(r)$ has poly-log diameter for $1 < r < 2$

We now present a natural generalization of our results in §3. We consider the diameter of $\mathcal{METR}(r)$, where $1 < r < 2$. Let $\mathcal{METR}(r,n)$ denote the subclass where the number of nodes is a given $n$.\(^{18}\)

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\(^{18}\)Paths found by decentralized routing where each step finds the best node (say, closest to the destination) in a neighborhood (using local topology). See [15, 19, 11].
Theorem 11. For $1 < r < 2$, for a $\cal{METR}(r,n)$ family, there exists a constant $c$ s.t., if the base graphs are $x_0$-strongly neighbored, where $x_0 = c \log \frac{n}{\varepsilon r}$, then almost surely the expected diameter of this family is upper bounded by a poly-log function.

For example, if we modify Kleinberg’s tree-based setting by connecting all the nodes together (say, order the nodes from left to right and connect them with undirected edges), and by using an inverse $r^{th}$ power distribution instead (for adding the random links), then this new setting has poly-log expected diameter. Note that, under the context of this section we define $d(u,v)$ as the cardinality of the smallest subtree which contains both $u$ and $v$.

Proof(sketch). We extend our approach in §3 (using probabilistic recurrence relations) to upper bound the diameter of the graphs in this abstract class. Define $\phi(x) = \max_{u \in V} \delta(B^u_x)$, where $\delta(B^u_x)$ is the diameter of the subgraph $B^u_x$ (induced in the base graph plus the random links), which is $\infty$ if the graph is not strongly connected. The non-decreasing $\phi(x)$ (and $\delta(B^u_x)$) is determined for each graph (with known random links).

The key idea is to establish a probabilistic recurrence relation, where $\phi(x) \leq 2\phi(x^{\xi}) + 1$ with probability $\geq 1 - n^{-2}$, for some chosen $0 < \xi < 1$ and for all $x$ greater than some chosen constant $x_0 > 0$. Then, we can just repeat theorem 1’s work to upper bound the expected diameter of $\cal{METR}(r)$.

For simplicity, we assume using metric functions ($\alpha = 1$). Let $r/2 < \xi < 1, x > 0$; $u, v$ are two arbitrary nodes with $d(u,v) = x$. Consider $Pr[B_u \to B_v]$, the probability of having a random link from $B_u = B_{x^\xi}(u)$ to $B_v = B_{x^\xi}(v)$. Using lemma 1, $Pr[B_u \to B_v] \geq 1 - e^{-\varepsilon x^\xi} - x^\xi$, where $\varepsilon = \varepsilon(B_u, B_v) = C_u(x + 2x^{\xi})^{-r} = \theta(x^{-r})$ and both $|B_u|$ and $|B_v|$ are $\theta(x^{\xi})$. Note that, for $1 < r$, the normalized coefficient $C_u$ is constant-bounded.

Now for any node $u \in V$, a ball $B_x(u)$ will have its diameter upper bounded by $2\phi(x^{\xi}) + 1$ if the events $B_v \to B_w$ occur for any two distinct nodes $v, w \in B_x(u)$ and $B_v \cap B_w = \emptyset$. The number of such events is $\leq n^2$, so using lemma 2, $Pr[\delta(B_x^u) \leq 2\phi(x^{\xi}) + 1] \geq 1 - n^2 \times e^{-\varepsilon x^{\xi r}}$. Combining the same event for all $u \in V$, $Pr[\phi(x) \leq 2\phi(x^{\xi}) + 1] \geq 1 - n^3 \times e^{-\varepsilon x^{\xi r}}$.

Now we choose $x_0 = (5/c)^{1/2 - r} \times \ln \frac{1}{\varepsilon r}$ and hence, for $x \geq x_0$, $Pr[\phi(x) \leq 2\phi(x^{\xi}) + 1] \geq 1 - n^{-2}$. Now simply choose $\xi = .5 + .25r$ (so, $r < 2\xi = (2 + r)/2 < 2$) and continue as in Theorem 1’s proof. □

References


Appendix

Proof of Lemma 5. Before proving this lemmna, we need to introduce some new notions. Let $Z(u,C)$ denote the indicator random variable, which indicates if the result $v = R_x(u)$ is outside of $C$ or not. For a fixed $u$, $E[Z(u,C)]$ is a function of $C$ on domain $2^V$. Consider all subsets $C$ with $|C| = n^\mu$ and corresponding values $E[Z(u,C)]$. Let $C_u$ denote the one that has $E[Z(u,C)]$ being the minimum. Let $Z_u$ denote $Z(u,C_u)$ and note that $E[Z_u] = \min\{E[Z(u,C)]| |C| \leq n^\mu\}$. Also note that the $Z_u$s for all $u$ are independent (though not always identical) Bernoulli random variables and from the definition of class $\mu$-Expansion each of them has expectation at least $\xi$.

Lemma 5 can be proved by using Chernoff’s inequality. Let $S = \{u_1, u_2, \ldots, u_l\}$ where $l = \lceil S \rceil = \Omega(\log n)$, and $T$ denote $\chi(S)-C$. If we produce an ordered sequence of $dl$ results $\chi_1(u_1), \chi_2(u_1), \ldots, \chi_q(u_1), \chi_1(u_2), \chi_2(u_2), \ldots, \chi_q(u_2), \ldots, \chi_1(u_k), \chi_2(u_k), \ldots, \chi_q(u_k)$ then $\chi(S)$ is the union of these $dl$ items. Therefore, $T$ can be seen as the result of accumulating only the fresh ones from these $dl$ items if they come one-by-one in order. A value $\chi_i(u_k)$ is seen fresh if it is neither in $C$ nor already in the current accumulated set $T$.

For $i = 1, q, k = 1, l$, let $X_k^i$ denote a random variable that takes value 1 if $\chi_i(u_k)$ is fresh, i.e. it is outside of $\bigcup_{j=1}^{i-1} \chi_j(u_k) \cup \bigcup_{j=1}^{q-1} \chi_j(u_k) \cup C$ (which clearly has size less than $(q + 1)|C| < n^\mu$ for large enough $n$). Otherwise, $X_k^i$ takes value 0. Let $X$ denote the sum of these $dl$ random variables. Note that each $X_k^i$ can be lower bounded by $Z_{u_k}$; thus $X$ can be lower bounded by the sum of $dl$ independent Bernoulli random variables, each of which has expectation at least $\xi = \beta/q$. Therefore, by applying Chernoff’s inequality we have $P_r[X \leq \beta(l(1 - \delta))] \leq e^{-\delta^2(\beta)/2}$ where $E[X] \geq (\log(\beta)/q) = \beta$ and $0 < \delta < 1$. Since $l = \Omega(\log n)$ then $e^{-\delta^2(\beta)/2} = O(n^{-\epsilon})$ for some $t = t(\delta) > 0$. This is due to a simple fact that $e^{\alpha \log n} = n^\alpha$ for any $\alpha$. Thus, $P_r[X \leq \beta(l(1 - \delta))] = O(n^{-\epsilon})$ or $P_r[X > l\beta(1 - \delta)] = 1 - O(n^{-\epsilon})$. This assures that when $n$ is large enough, $X$, and thus $|T|$, is almost surely greater than $l\beta(1 - \delta)$. Set $\gamma = \beta(1 - \delta)$.

When we have obtained this fixed $\delta$, given any $\theta > 0$ we can choose $c$ s.t. when $l > c \log n$ we have $P_r[X > l\beta(1 - \delta)] = 1 - O(n^{-\theta})$. This can be done by choosing $c$ s.t. $e^{-\delta^2(\beta c \log n)/2} = O(n^{-\theta})$ or $\delta^2/2 > \theta$ or $c > \frac{2^{\frac{\theta}{2}}}{\beta^2\gamma}$. Put another way, we can start choosing any $\gamma$ s.t. $1 < \gamma < \beta$ (thus we have the intermediate $\delta = 1 - \gamma/\beta$) and then choose any $c$ s.t. $c > \frac{2^{\frac{\theta}{2}}}{(\beta - \gamma)^2}$.