A stub of what could someday become a PGF tutorial

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$$f(x) = \sum_k a_k x^k \tag{1}$$

$$a_n = \frac{1}{n!} \left. \frac{d^n f(x)}{dx^n} \right|_{x=0} \tag{2}$$

$$a_n = \frac{1}{2\pi i} \oint \frac{f(z)}{z^{n+1}} dz = r^{-n} \int_0^1 \mathrm{e}^{-2\pi i n \theta} f\!\left(r\,\mathrm{e}^{2\pi i \theta}\right) d\theta \tag{3}$$

We could approximate this integral by evaluating $f_m = f\!\left(r\,\mathrm{e}^{2\pi i \frac{m}{M}}\right)$ along the M equally spaced points $\{f_0, f_1, \ldots, f_{M-1}\}$.

$$a_n \approx \frac{1}{M r^n} \sum_{m=0}^{M-1} f_m\,\mathrm{e}^{-2\pi i n \frac{m}{M}} \tag{4}$$

Since the f_m do not depend on n, the same points could be used in order to evaluate different a_n. In fact, the sum happens to be a discrete Fourier transform (DFT).

$$\{M a_0, M r a_1, M r^2 a_2, \ldots, M r^{M-1} a_{M-1}\} \approx \mathcal{F}^{-1}\{f_0, f_1, f_2, \ldots, f_{M-1}\} \tag{5}$$

Hence, all the $\{a_0, a_1, \ldots, a_{M-1}\}$ may be evaluated in the same pass of fast Fourier transform (FFT) algorithm.[1]

If $f(x)$ is a polynomial of order N such that $N < M$, this relation becomes *exact*: the discrete sum exactly evaluates the integral in the limit $M \to \infty$ and, by the Nyquist-Shannon sampling theorem, the result of the DFT for $0 \leq n \leq N$ should not depend on M when $N < M$. (Hence, a finite M greater than N gives the same result as $M \to \infty$, which is exact.)

When $N \geq M$, aliasing occur: the a_n for $n \geq M$ are “folded back” unto the lower values of n, resulting in errors. However, even when $a_n \neq 0$ in the limit $n \to \infty$, it is often possible to choose a M sufficiently high such that the error is acceptable, provided that $\sum_{n \geq M} a_n$ is small or that we have some idea of the behaviour of a_n for $n \geq M$.

[1] We here suppose that the inverse transform $\mathcal{F}^{-1}$ is defined without normalization, which is the case of the FFTW algorithm as well as most standard libraries. Note, however, that MATLAB applies a factor $\frac{1}{M}$.