Community Structure

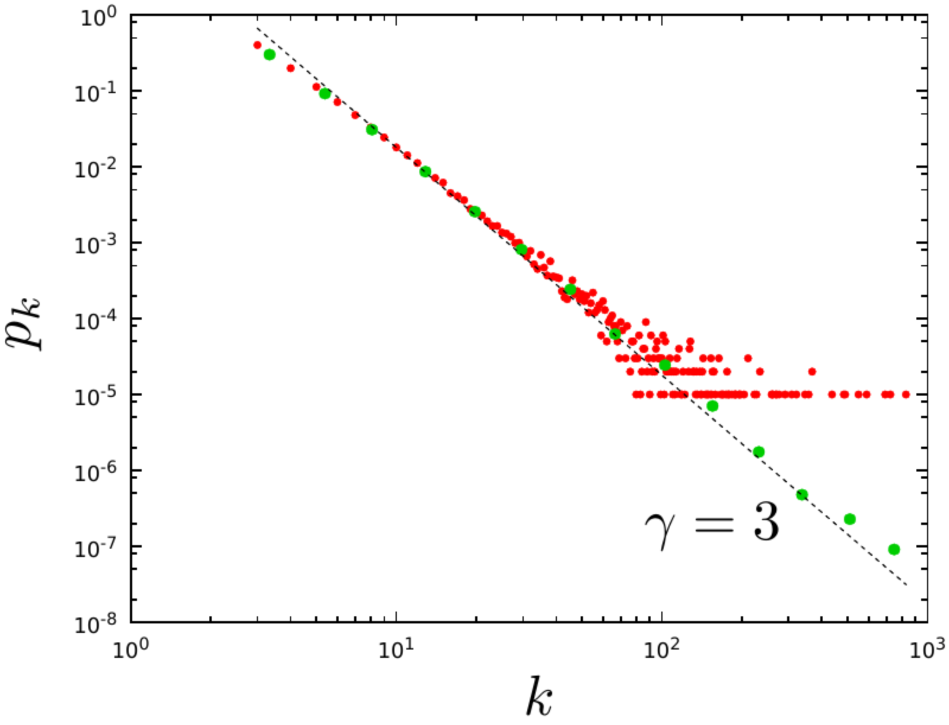
Network Theory and Applications

ECS 253 / MAE 253

Spring 2018

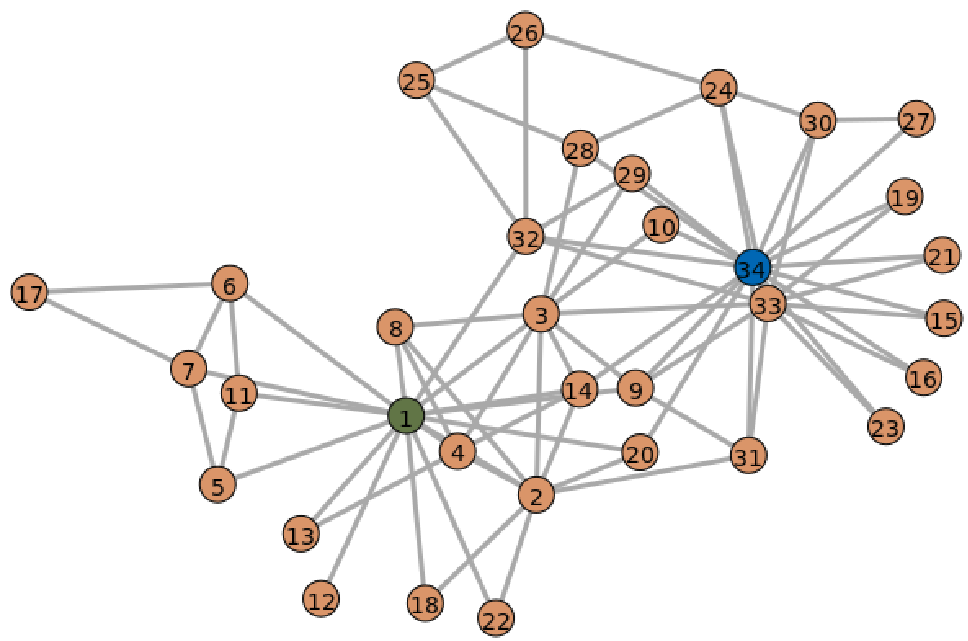
Communities

- So far:
  - Properties of nodes: degree, centralities, triangles
  - Statistics of local properties: degree distribution
- How about large-scale organization?
  - Core-periphery, hierarchy,...
  - <u>Communities</u>



Introduction – Zachary's karate club

- Social scientist Wayne W. Zachary collected friendship data at a karate club from 1970 to 72
- Edges represent friendship (activities outside the club)

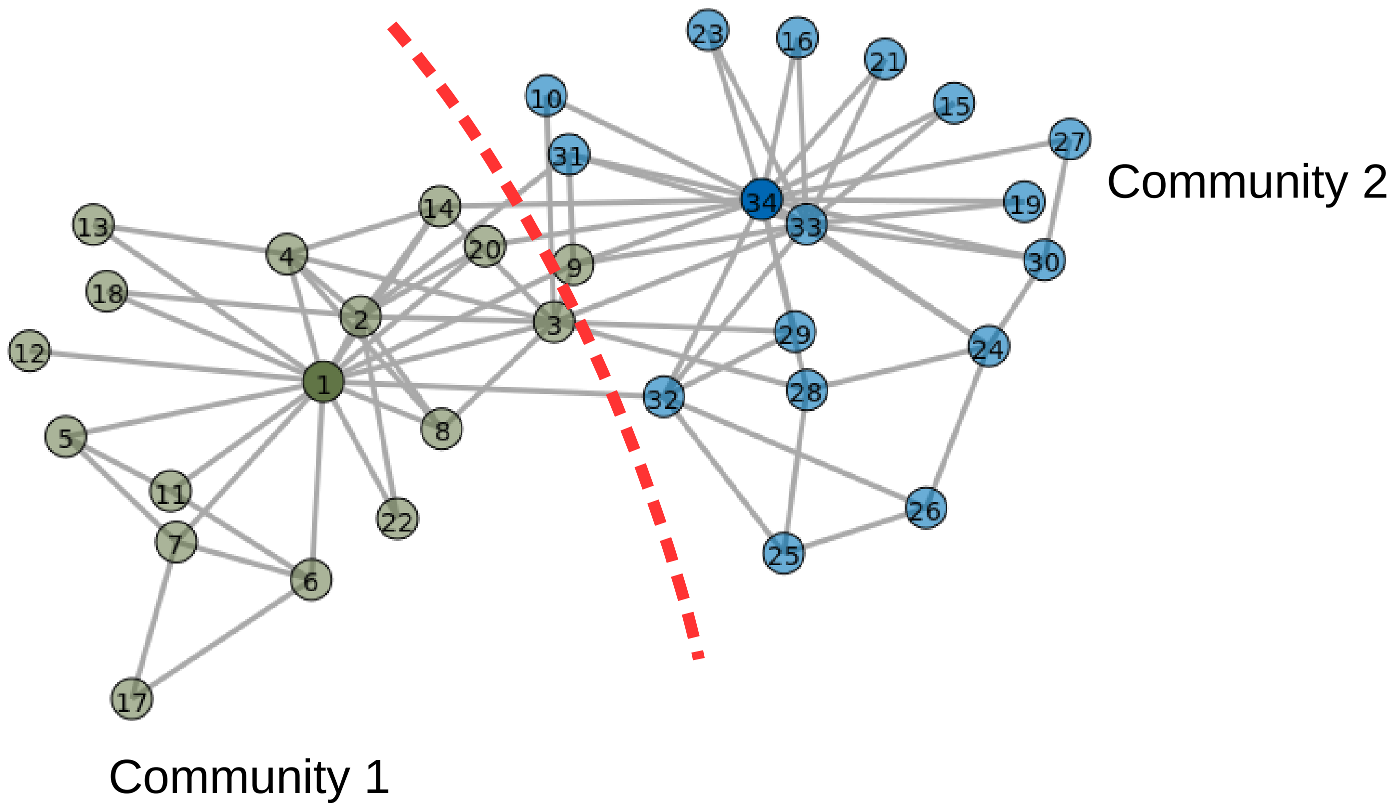


- Conflict between instructor (node 1) and administrator (node 2), group broke up into two
- Q: Can we predict the groups based on network structure?

Zachary, Wayne W. "An information flow model for conflict and fission in small groups." Journal of anthropological research 33.4 (1977): 452-473.

Introduction – Zachary's karate club

- Zachary's solution: network flow, source=node 1, sink=node 34
- Cut: flow bottleneck



- All but node 9 correct

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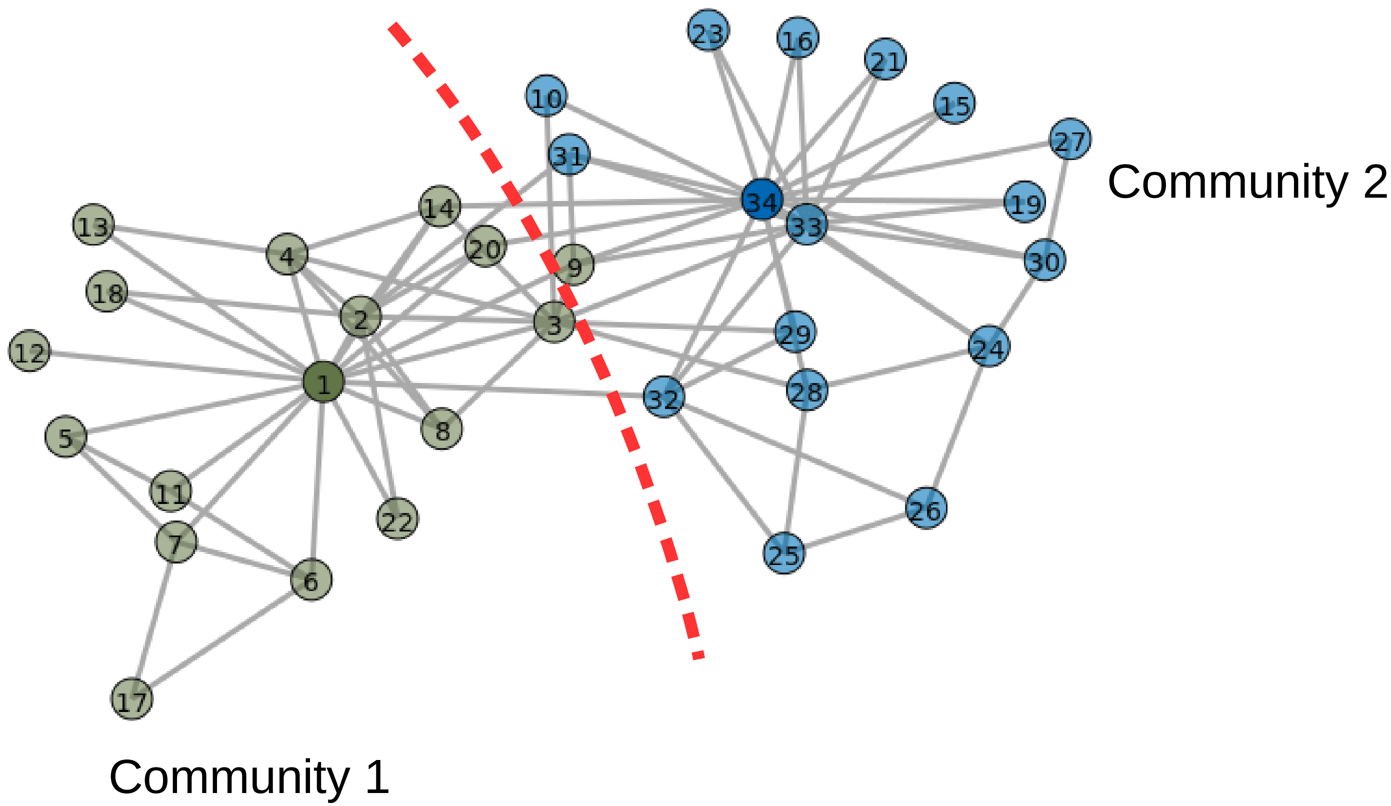

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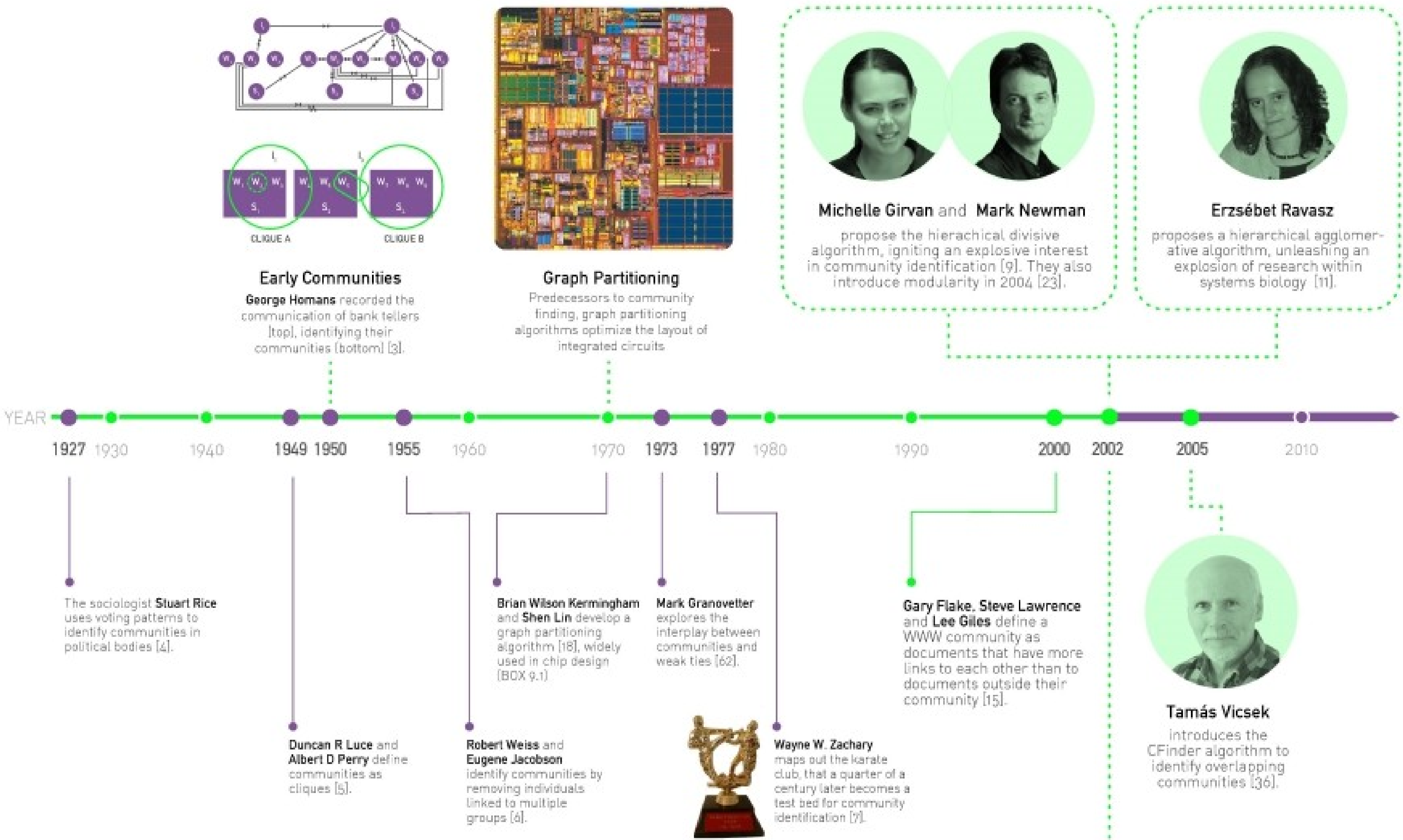
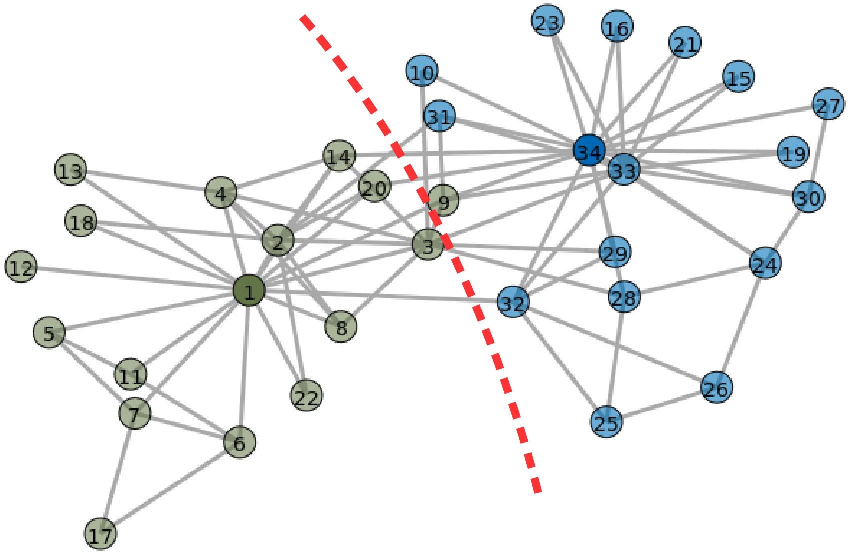
History

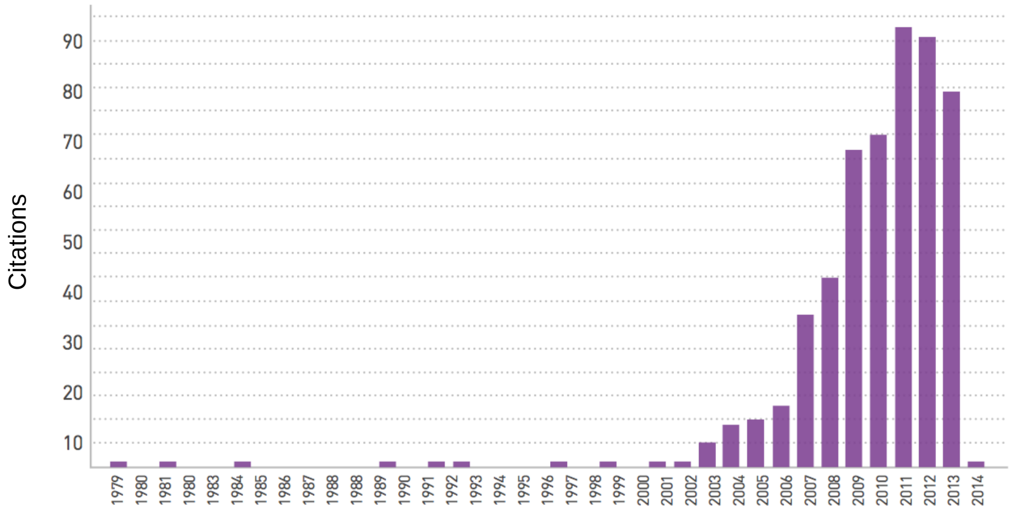


Barabási, Network Science

Introduction – Zachary’s karate club

- Communities = locally dense subgraphs
- Modern network community detection from 2000s





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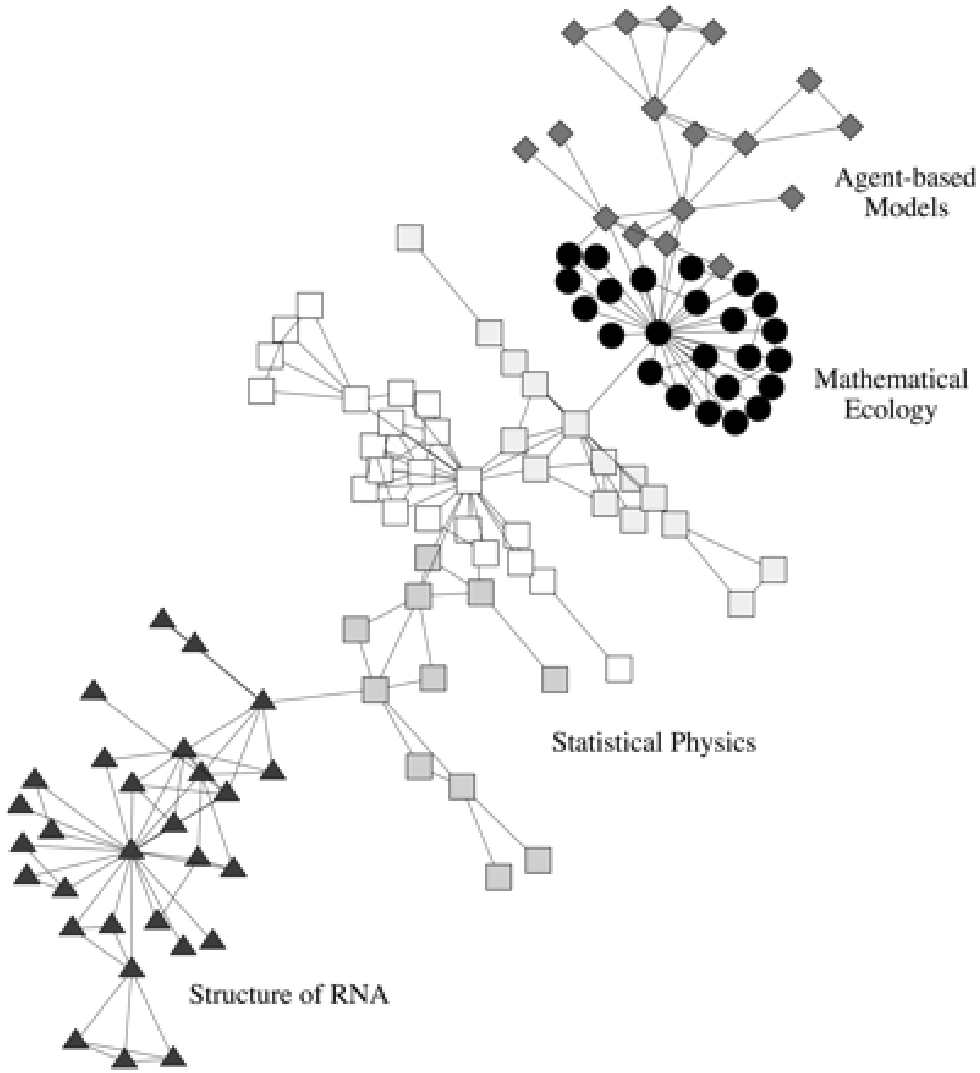
Zachary Karate Club Club

First presenter to mention the ZKC at a conference gets the trophy.

http://networkkarate.tumblr.com/

Scientific collaboration at SFI

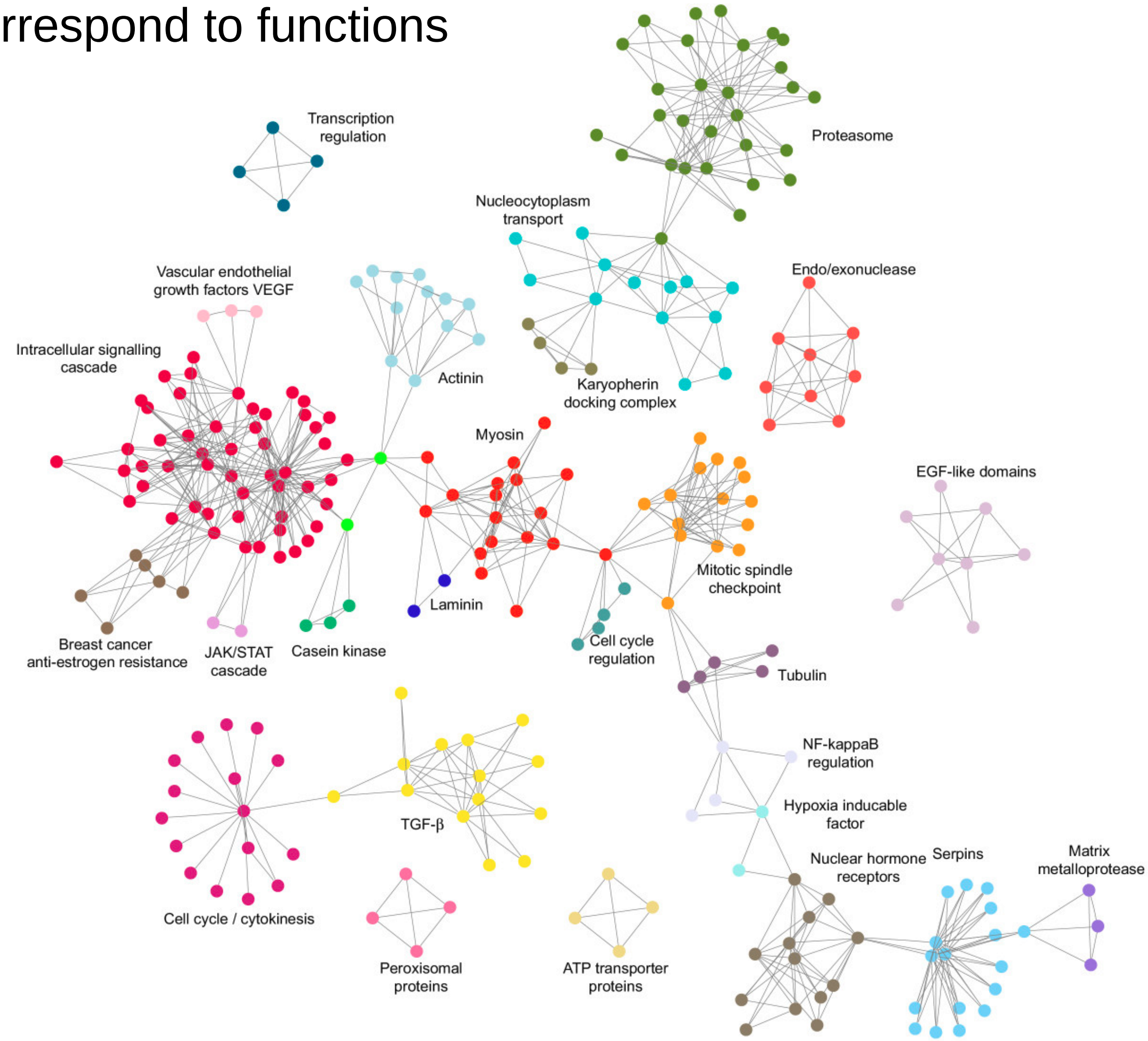
- Link (A – B) : A and B coauthored a paper
- Node classification by role according to position in community



M. Girvan et al., *PNAS*, 99:12 (2002)

Rat protein-protein interaction network

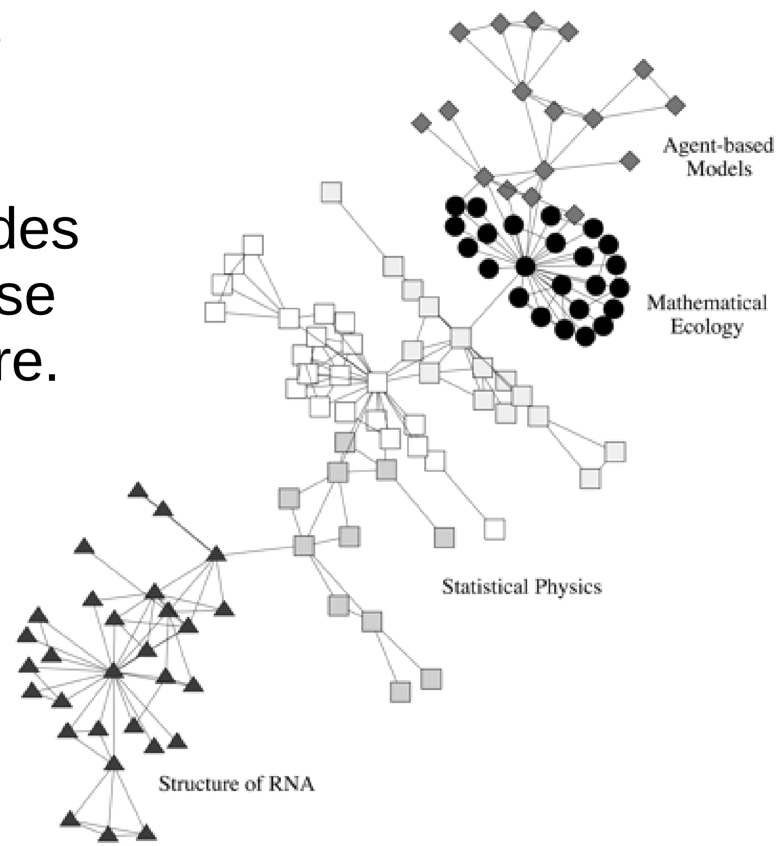
- Link (A – B) : A and B proteins physically interact
- Modules correspond to functions



PF Jonsson et al., *BMC Bioinf.* 7:2 (2006).

Basics

- What is a community?
  - Intuitively: densely connected subnetworks
- Why is it interesting?
  - Nodes that participate in same function/nodes with similar attributes form groups and these groups are represented in network structure.
- Challenges:
  - No single clear definition.
    → Many competing options.
  - Large networks, different features.
    → Even more algorithms.
  - Which one to choose?
  - How to evaluate performance of method?



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Outline

- How to identify communities?
- How to asses the quality of a community division?
- How asses the quality of a method?

- A lot of competing methods and measures

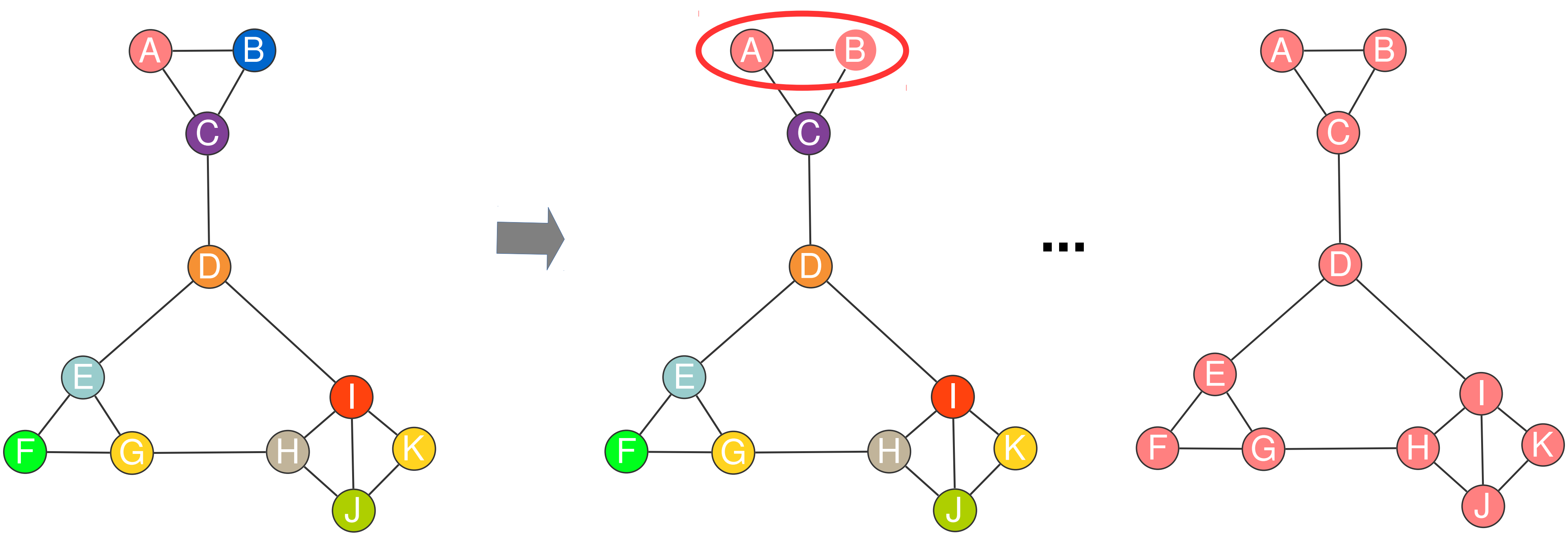
  → Here: selection that shows the development of the field
- And some guidelines to navigate the field

Method 1:
Hierarchical Clustering

Hierarchical clustering

- Traditional method used by social scientists.
- 0) Define a distance metric σ_{ij} between nodes based on network

1) Each node in its own community.
2) Calculate a distance between pairs of communities according to some rule.
3) Join closest pair.
4) Go to step 2.



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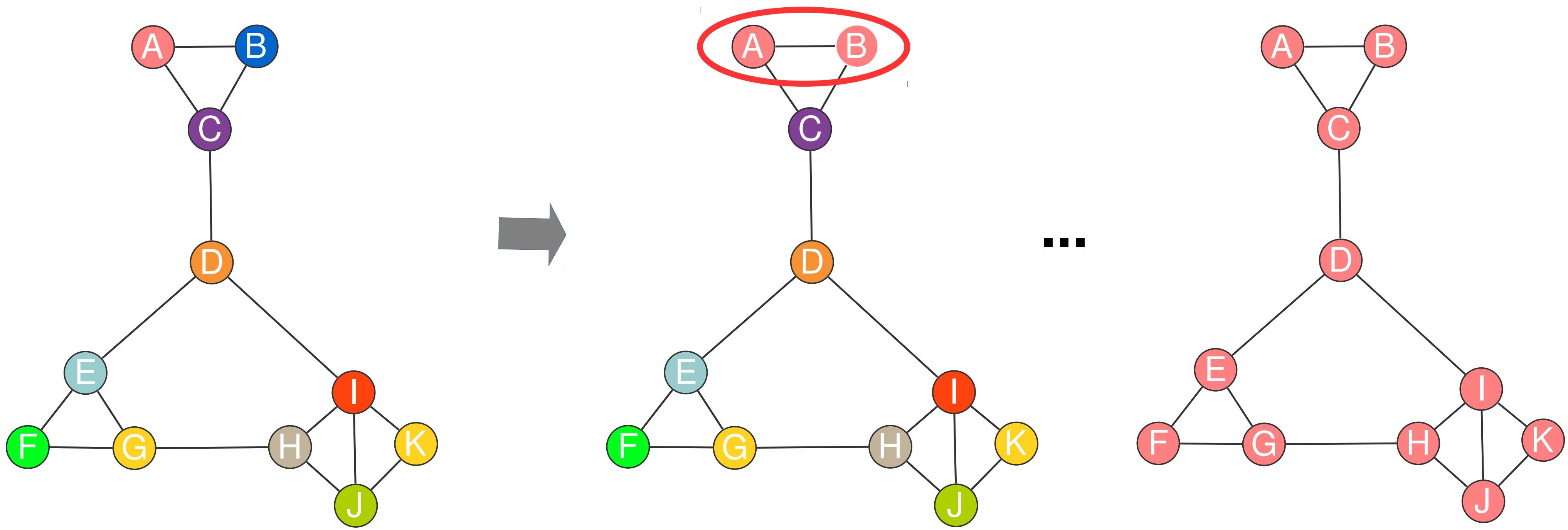


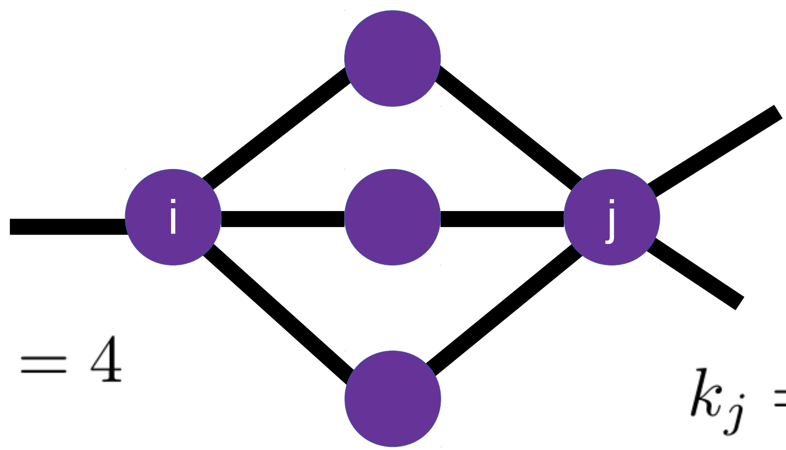
Hierarchical clustering: distance

- Node distance should be low if nodes are in a community.
- Popular choice:

#common neighbors

$$\sigma_{ij} = \frac{n_{ij}}{\sqrt{k_i k_j}}$$

degree



$$k_i = 4$$

$$k_j = 5$$

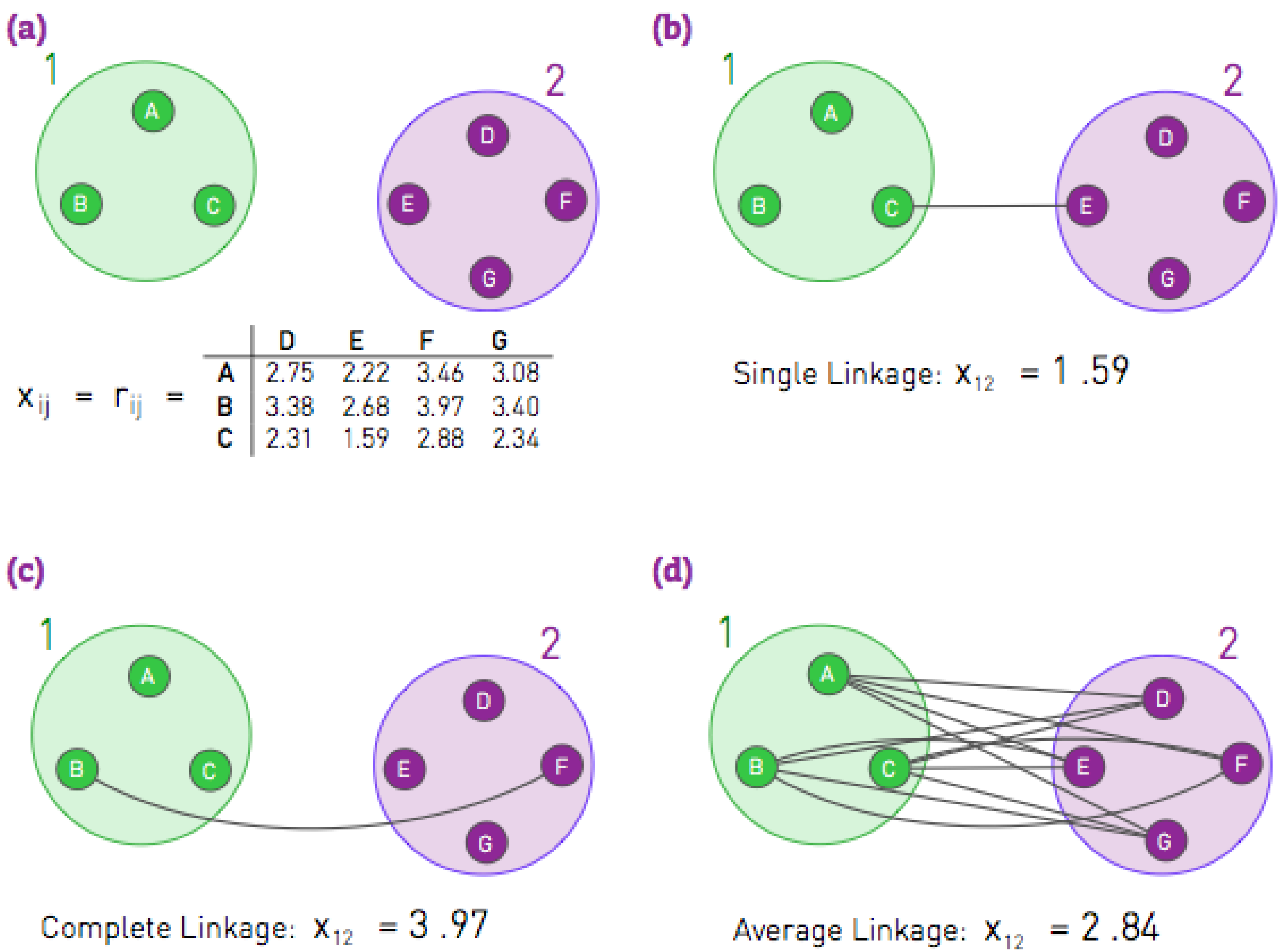
$$n_{ij} = 3$$

$$\sigma_{ij} = \frac{3}{\sqrt{4 \cdot 20}} \approx 0.67$$

- Other distances possible, e.g. number of independent paths connecting i and j

Hierarchical clustering: distance

- Distance between communities with more than one node:

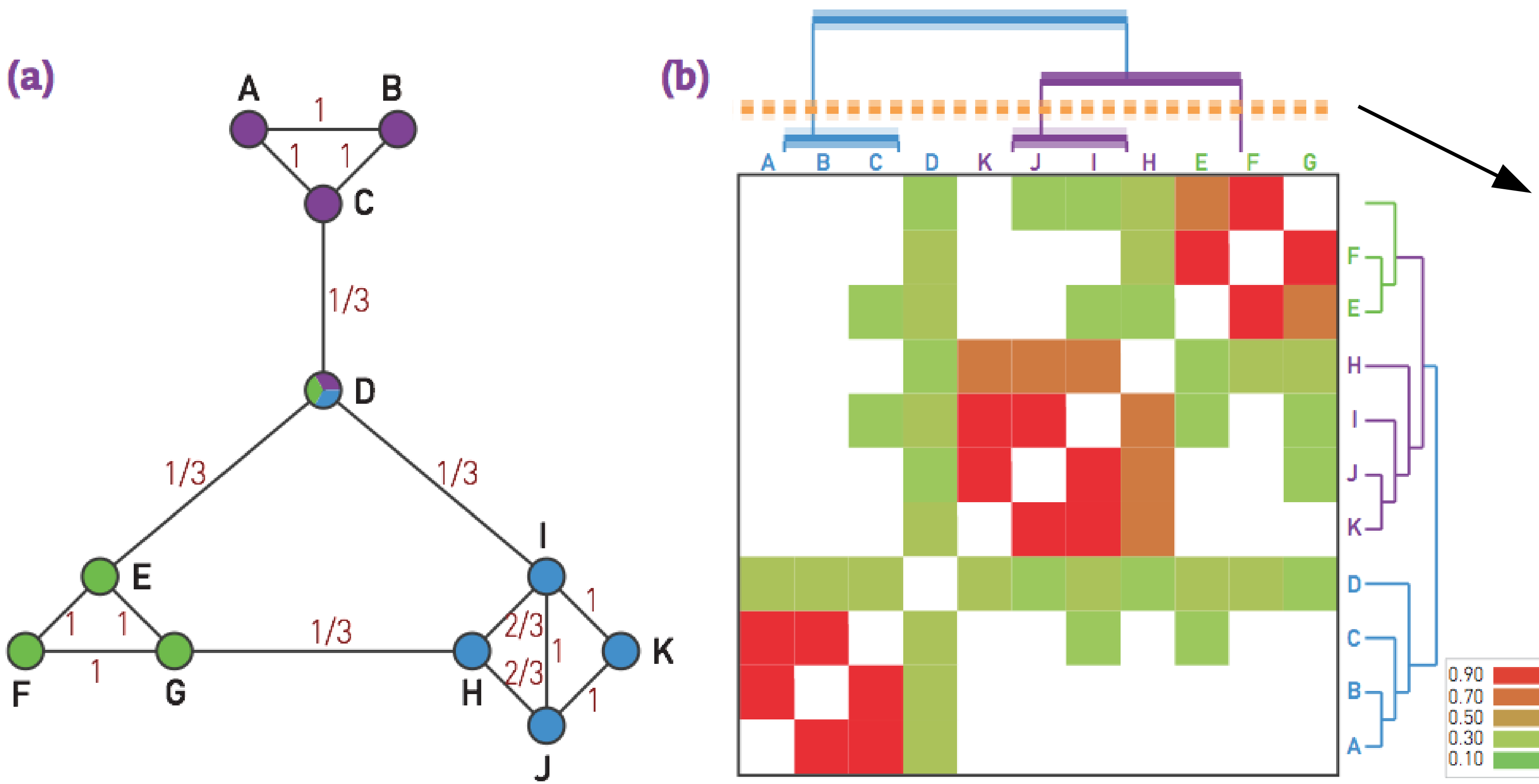


Hierarchical clustering: result

- Example artificial network

  (slightly different definition of distance)

Cut corresponds to one partition

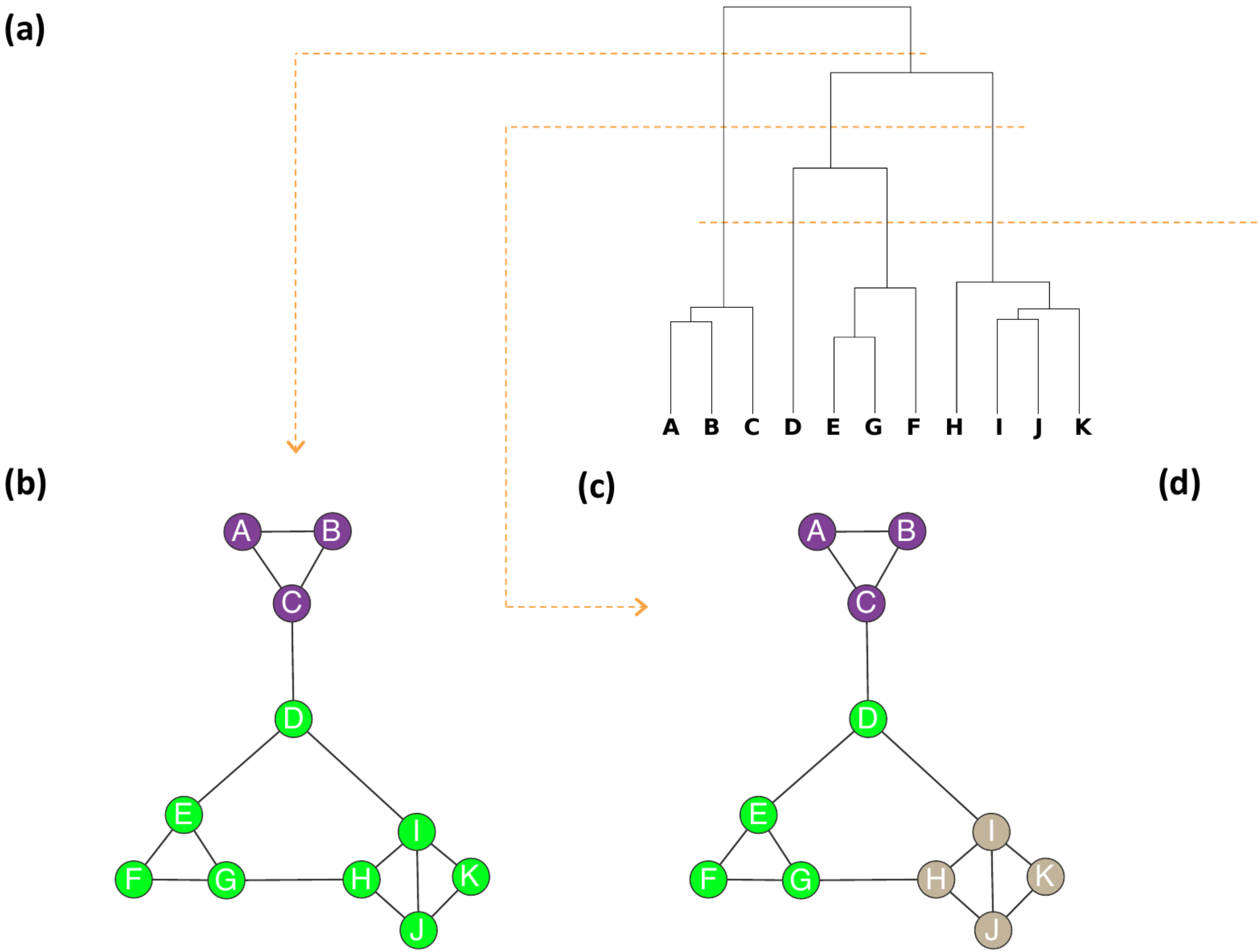


dendrogram

A.-L. Barabási, *Network Science: Communities*.
E. Ravasz et al., *Science* 297 (2002).

Hierarchical clustering: result

- Dendrogram



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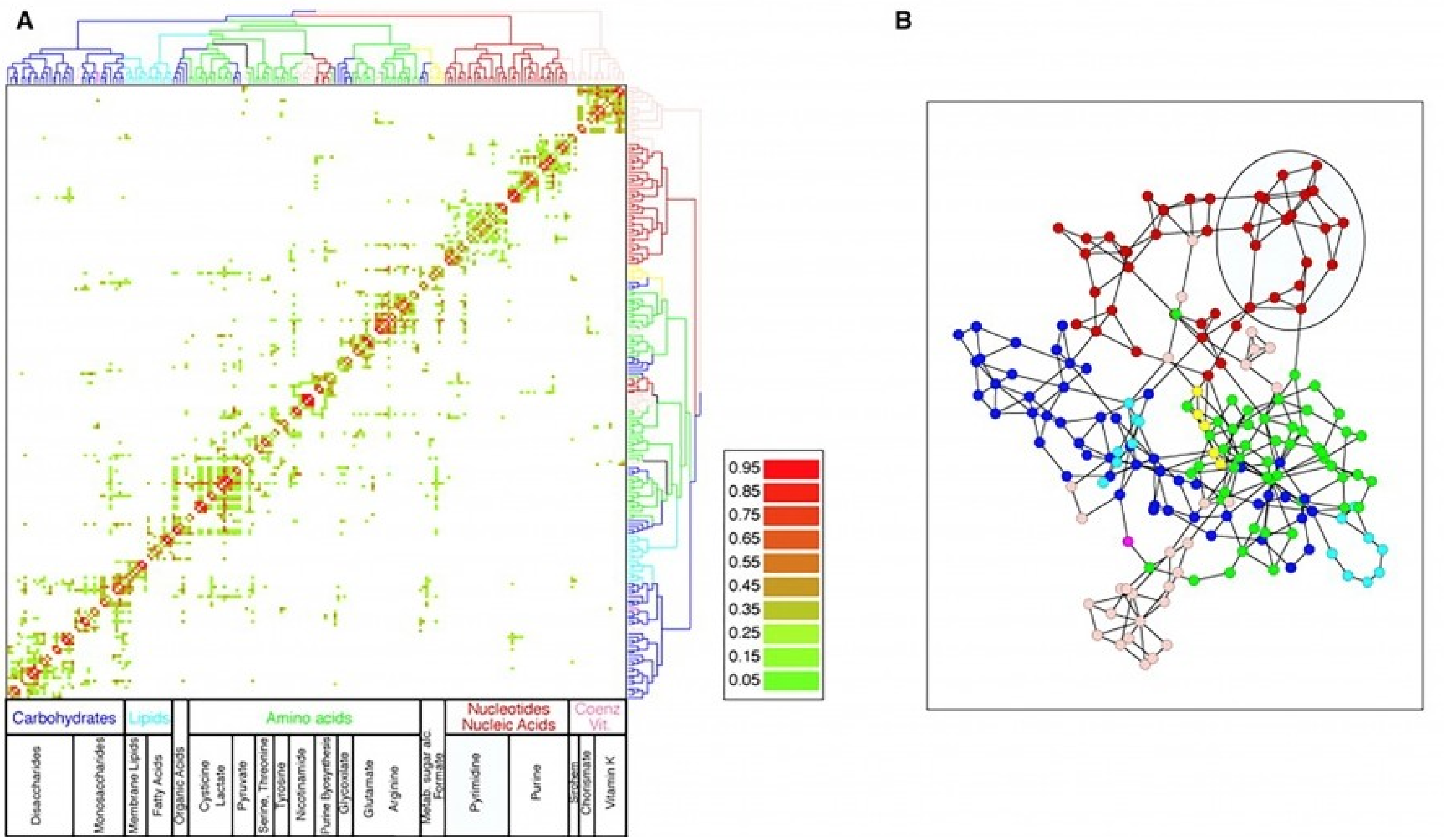
Hierarchical clustering: application

- E. coli metabolism



- The color of each node, capturing the predominant biochemical class to which it belongs, indicates that different functional classes are segregated in distinct network neighborhoods.
- The highlighted region selects the nodes that belong to the pyrimidine metabolism, one of the predicted communities.

A.-L. Barabási, *Network Science: Communities*.

Hierarchical clustering: issues

- Advantages:
  - Easy to understand
  - Easy to implement
- Disadvantages:
  - Slow(ish), number of steps to evaluate: $\sim N^2$ or $\sim N^3$, depending on linkage
  - “Tends to group together those nodes with the strongest connections but leave out those with weaker connections → divisions consist of a few dense cores surrounded by a periphery of unattached nodes”
  - (Results depend on distance and linkage)
- Open question: where to cut the dendrogram?

M.E.J. Newman, *Nature Phys.*, 8:25-31 (2012)

Method 2:
Betweenness based division

Betweenness based division

- Alternative method: instead of agglomerating communities, breaking one large into smaller ones

1) Start from one large community
2) Calculate a centrality measure for each link
3) Remove link with highest centrality
4) Go to step 2

- Centralities: **betweenness for** edge *i*

$$B(i) = \sum_{s \neq t} \frac{\sigma_{st}(i)}{\sigma_{st}}$$

#shorest path between s and t traversing i

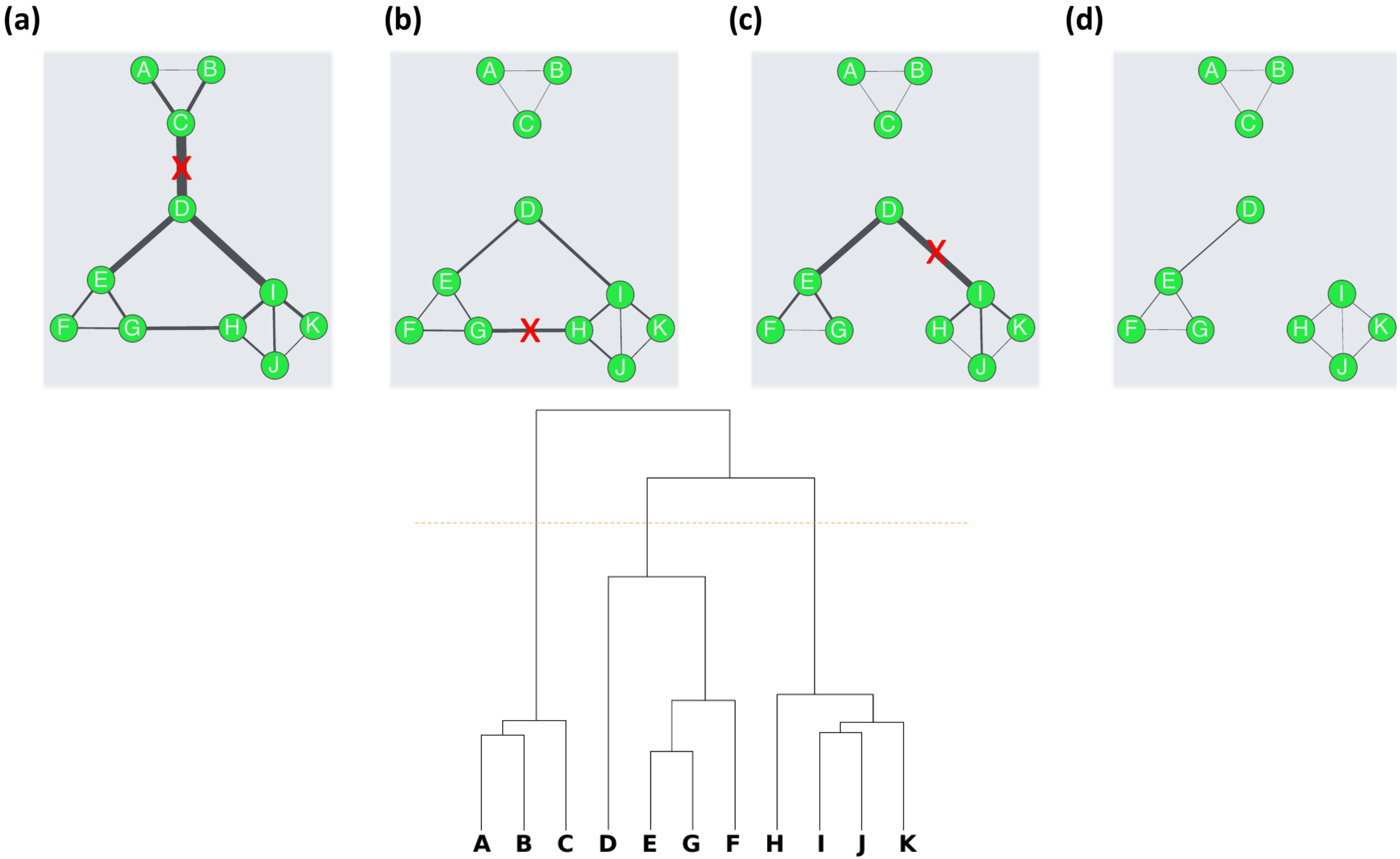
#shorest path between s and t

- Other: random walk, ...

M. Girvan et al., *PNAS*, 99:12 (2002)

Betweenness based division

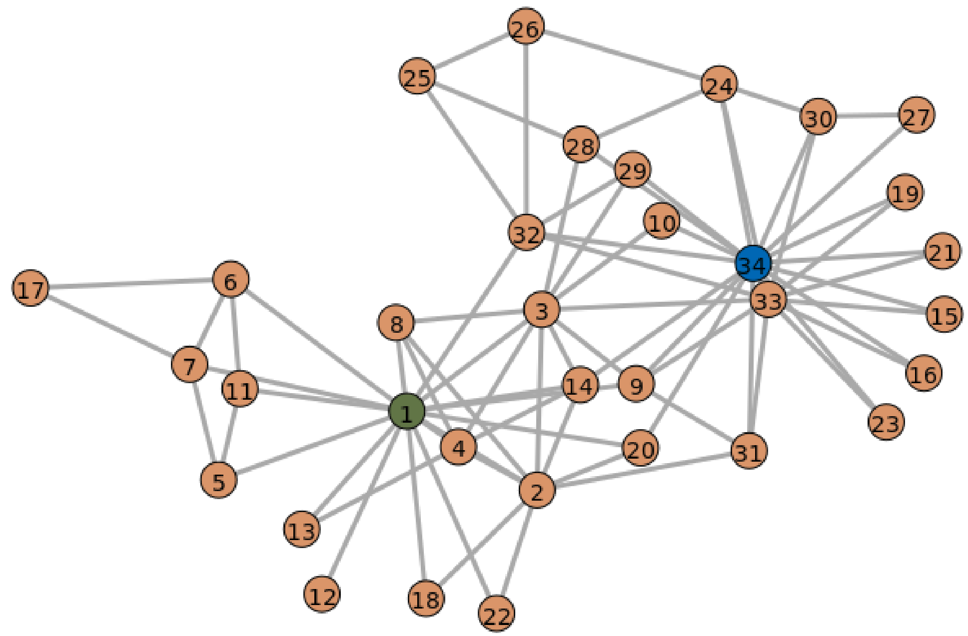
- Our little example:

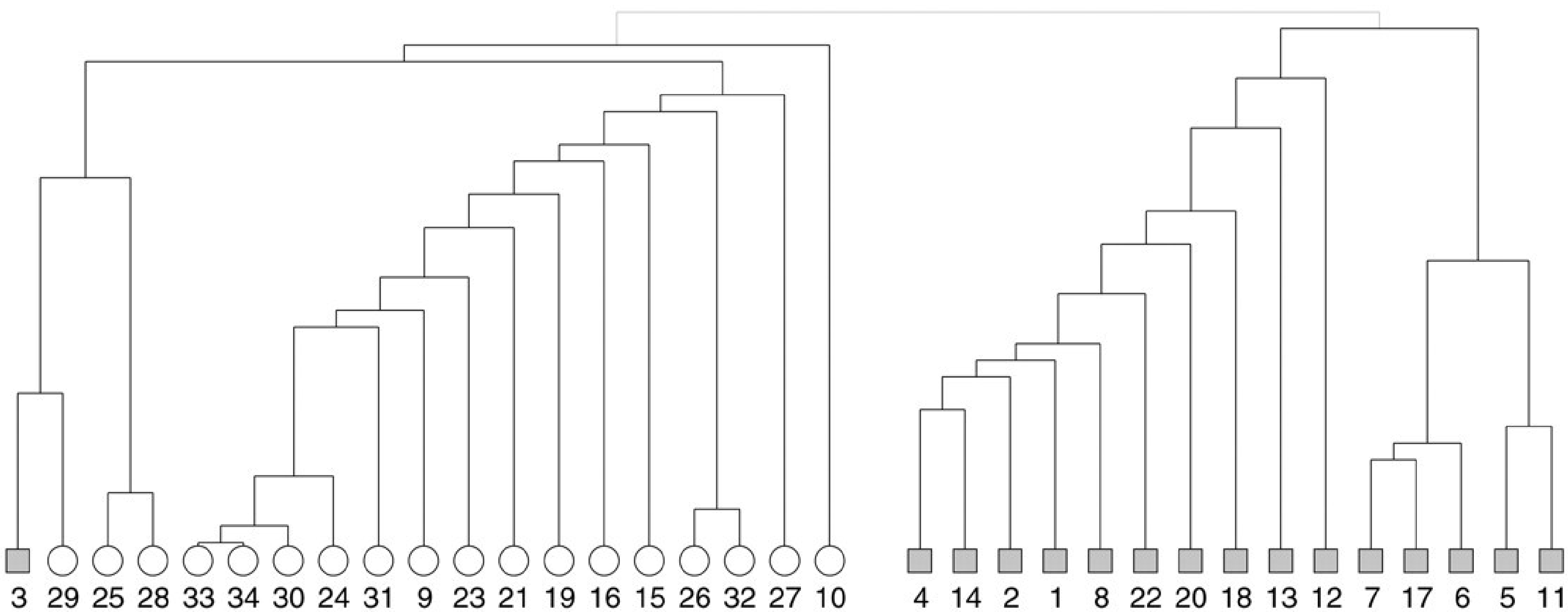


A.-L. Barabási, *Network Science: Communities*.

Betweenness based division: application

- Zachary’s Karate Club





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Betweenness based division

- Advantages:
  - Easy to understand
  - Easy to implement
  - Perhaps less decisions have to be made
- Disadvantages:
  - Slow, number of steps to evaluate: ~NL^2 or faster if we don't recalculate the betweenness in each step
  - (Results depend on centrality)
- **Still open question: where to cut the dendrogram?**

M.E.J. Newman, *Nature Phys.*, 8:25-31 (2012)

Quality of community division

Modularity

- Again: many existing measures.
- Naïve: fraction of links that are inside communities

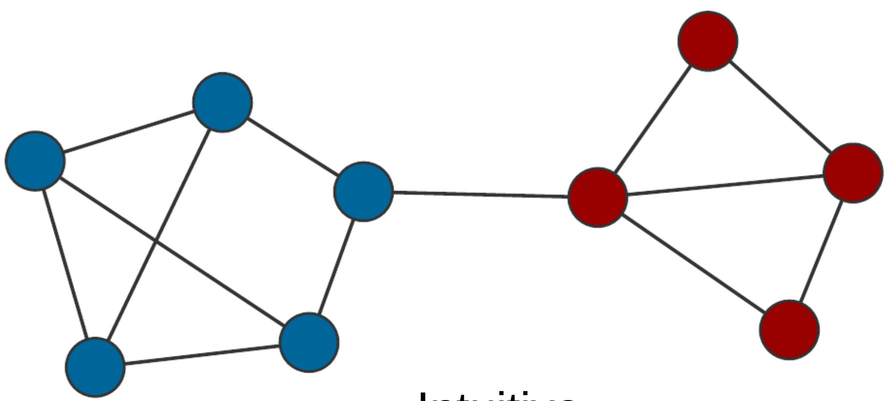
$$Q = \frac{1}{2m} \sum_{ij} A_{ij} \, \delta_{s_i, s_j}$$

- m: #links in the network
- A_{ij}: adjacency matrix, 1 if i and j are connected, 0 of not
- δ_{s1s2}: 1 if in the same community, 0 if not

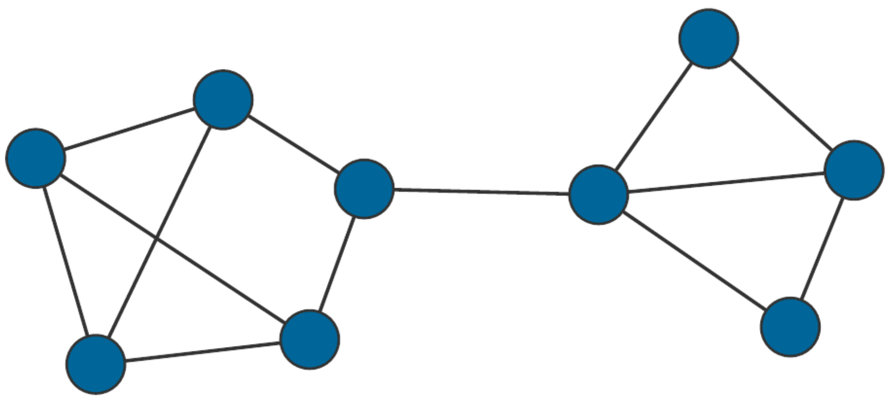
M. Girvan et al., *PNAS*, 99:12 (2002)

Modularity

$$Q = \frac{1}{2m} \sum_{ij} A_{ij}\, \delta_{s_i, s_j}$$



Intuitive.



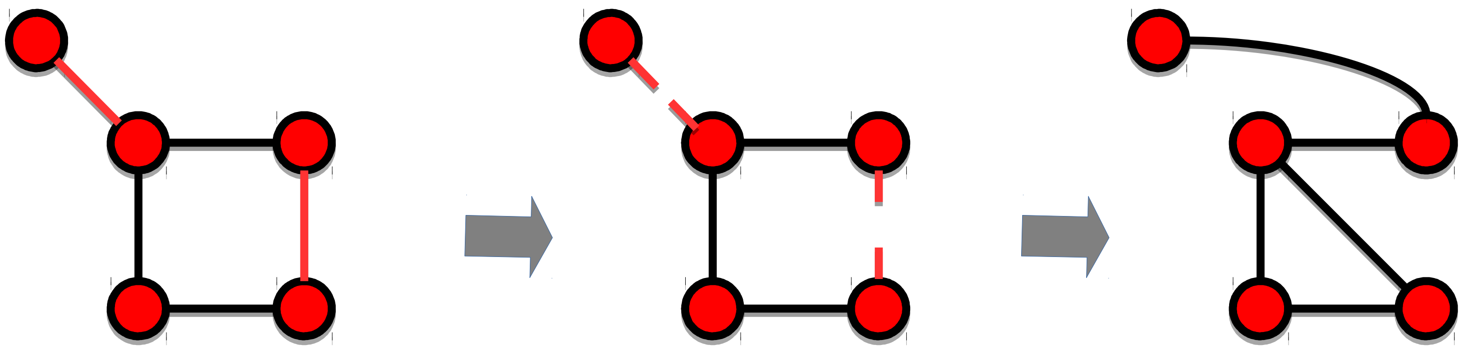
Optimal.

- Entire network one community: even if links are randomly placed, all links are inside the community
- Instead: fraction of links inside communities compared to what you would expect by chance
- What is by chance?

M. Girvan et al., *PNAS*, 99:12 (2002)

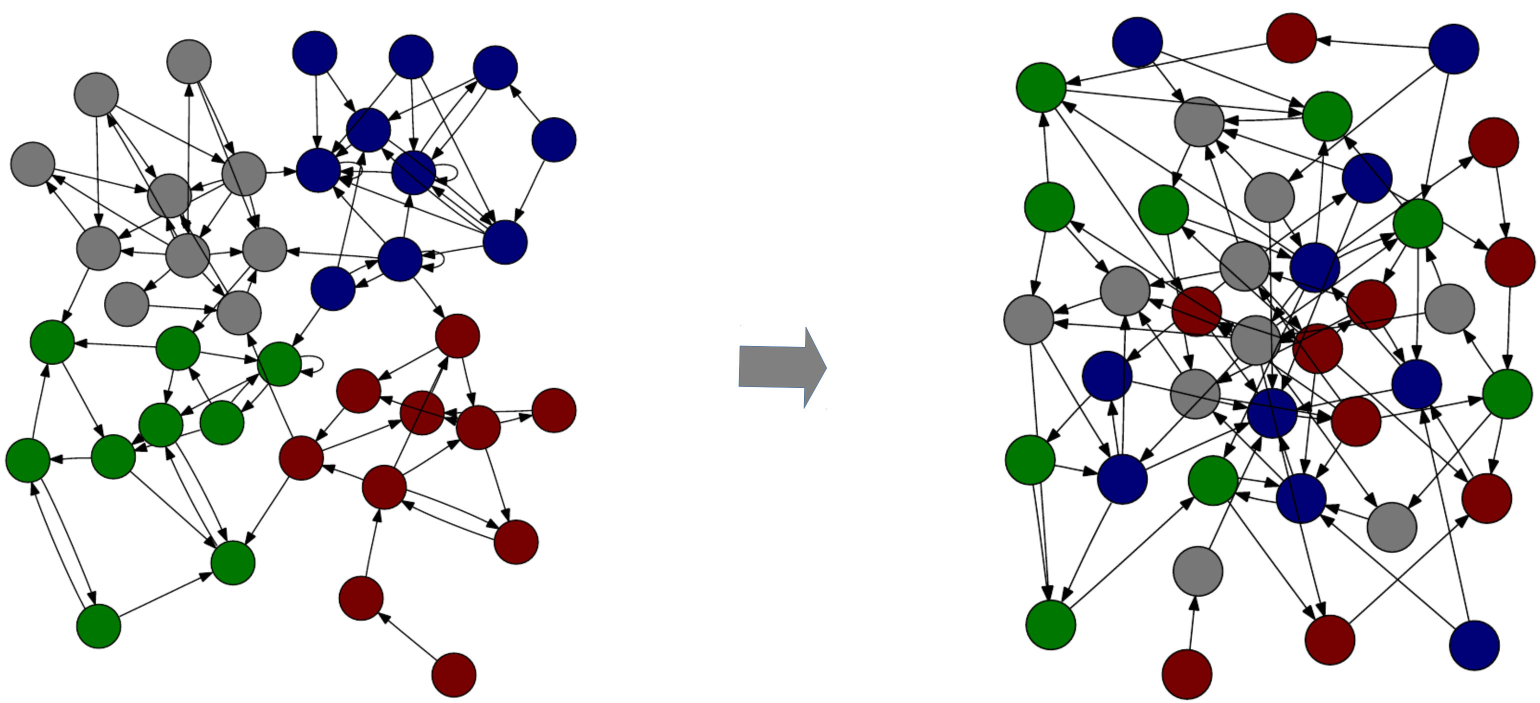
What is by chance?

- Degree preserved randomization



Original

What we compare to



Modularity

- Modularity:

$$M = Q = \frac{1}{2m}\sum_{ij}\left[A_{ij} - \frac{k_i k_j}{2m}\right]\delta_{s_i,s_j}$$

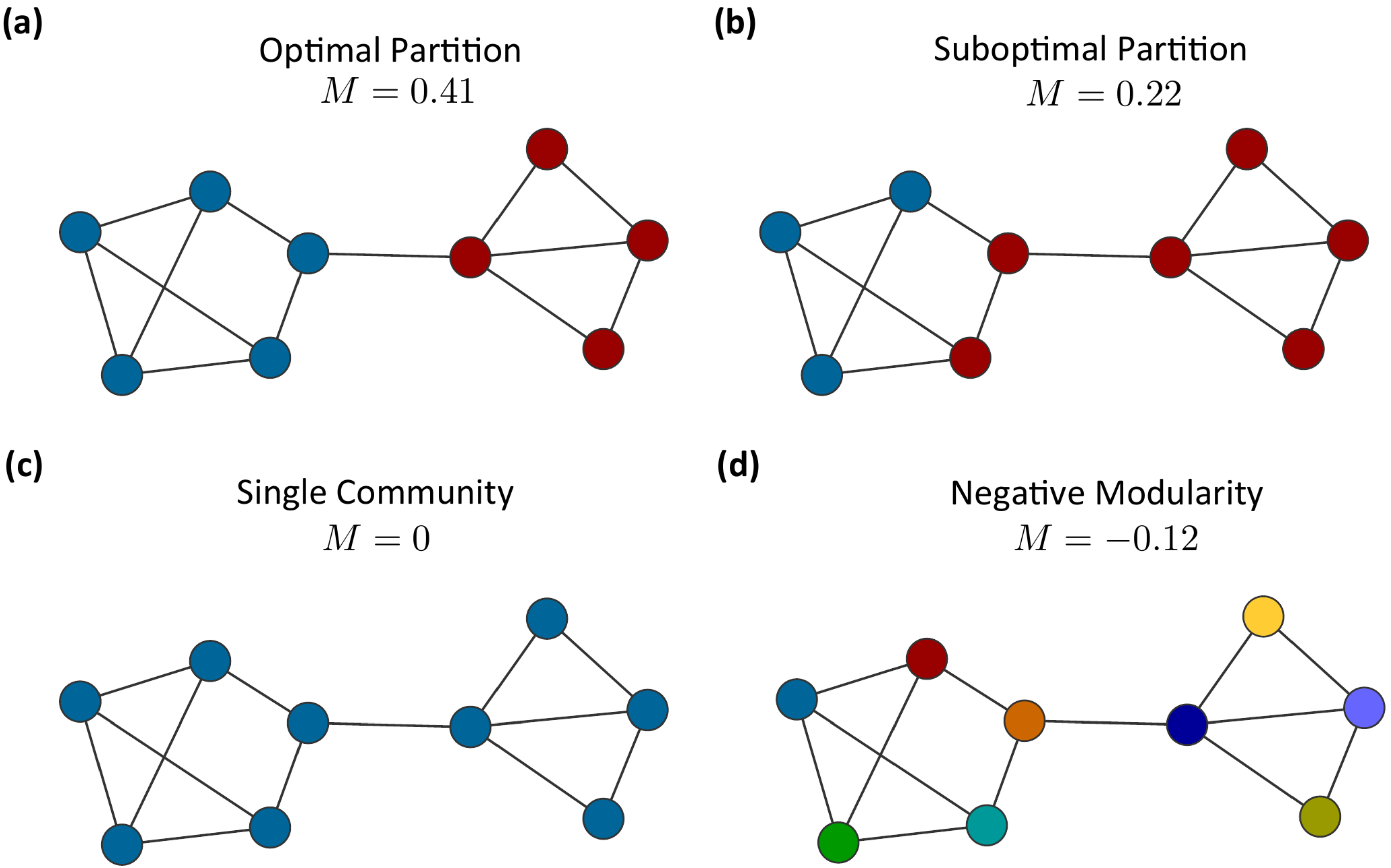
Real link

Probability of link in randomized version

- m: #links in the network
- A_{ij}: adjacency matrix, 1 if i and j are connected, 0 of not
- k_i: degree of node I
- δ_{s1s2}: 1 if in the same community, 0 if not
- High M → good division

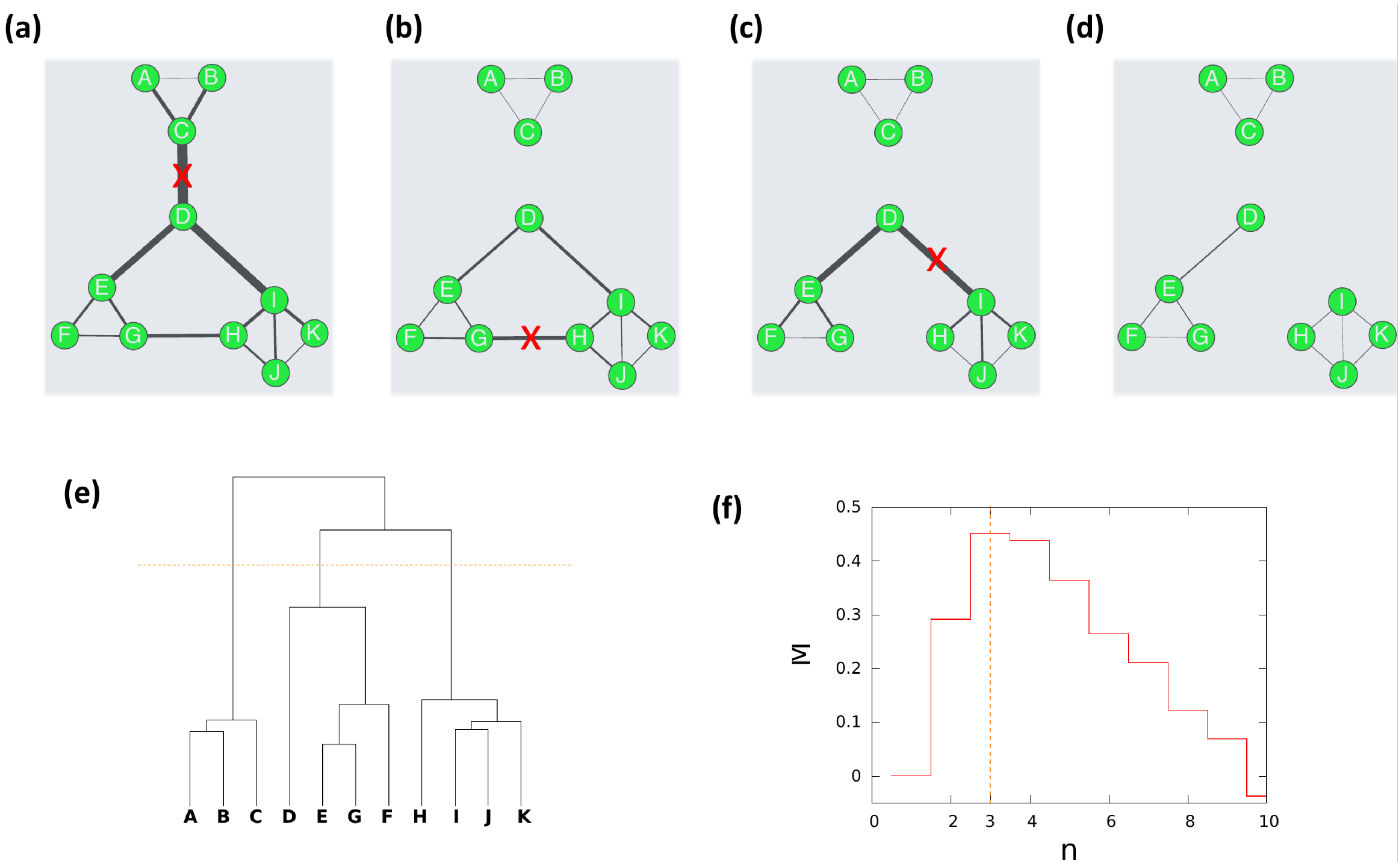
M. Girvan et al., *PNAS*, 99:12 (2002)

Modularity


(a)
Optimal Partition
$M = 0.41$
(b)
Suboptimal Partition
$M = 0.22$
(c)
Single Community
$M = 0$
(d)
Negative Modularity
$M = -0.12$

Modularity

- Where to cut dendrogram? At maximum Q!



M. Girvan et al., *PNAS*, 99:12 (2002)

Method 3:
Direct optimization of Q

Direct optimization of modularity

- Exact maximum of Q → NP-complete (exponentially increasingly difficult with N)
- Approximation methods: a lot to choose from
- Louvain method
  - Fast, ~L
  - Typically preforms well on tests
- Two steps applied iteratively:

1) Find local maximum.

2) Coarse grain network.

V.D. Blondel et al, *J. Stat. Mech*. P10008 (2008).

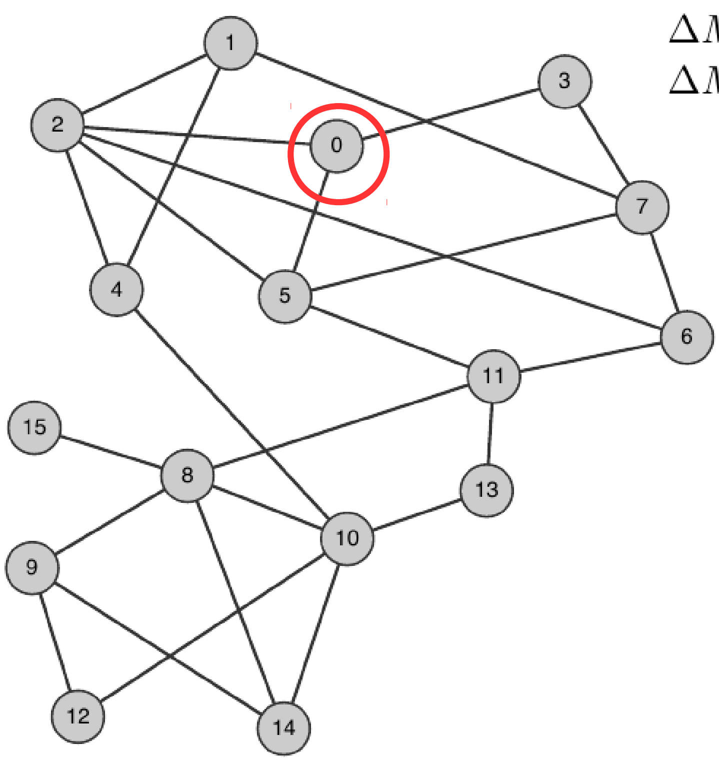
Louvain method: Step 1

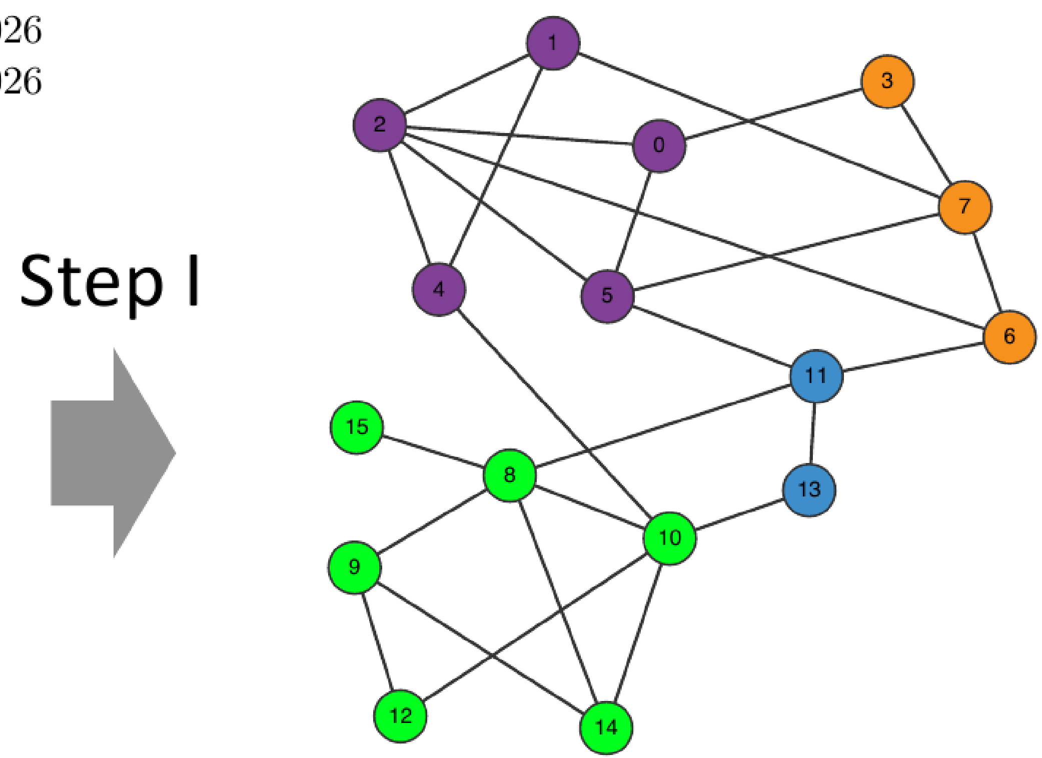
1) For node i, calculate ΔQ for each neighbor j if node i is removed from its community and placed in the commnuity of j.Coarse grain network.

2) Move i to community that maximizes ΔQ, if ΔQ>0.

3) Repeat while Q increases.

$$\Delta M_{0,2} = 0.023$$
$$\Delta M_{0,3} = 0.032$$
$$\Delta M_{0,4} = 0.026$$
$$\Delta M_{0,5} = 0.026$$





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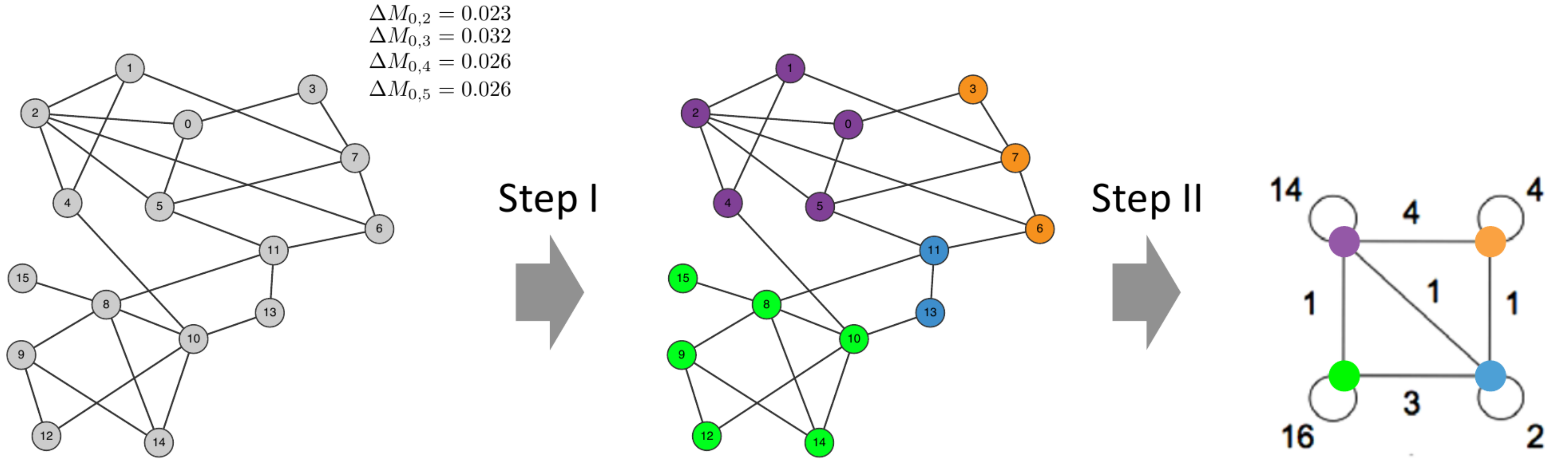
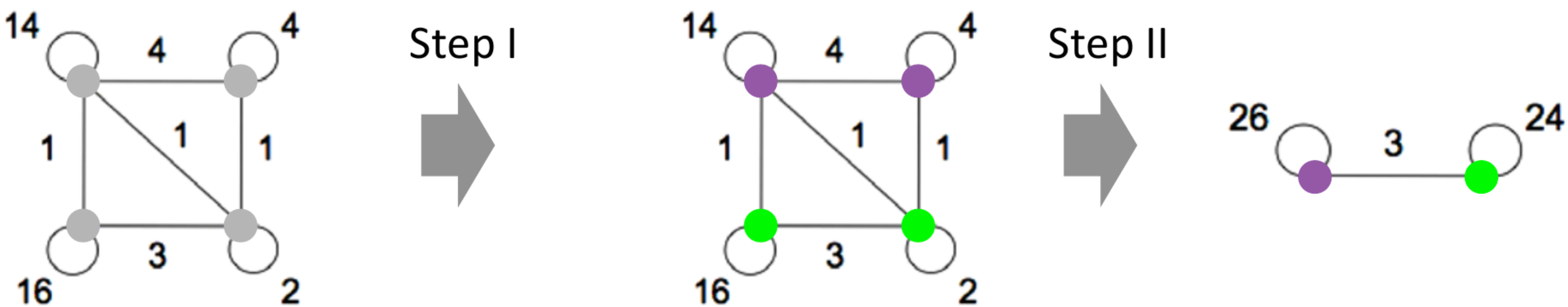
Louvain method

1st Pass



2nd Pass



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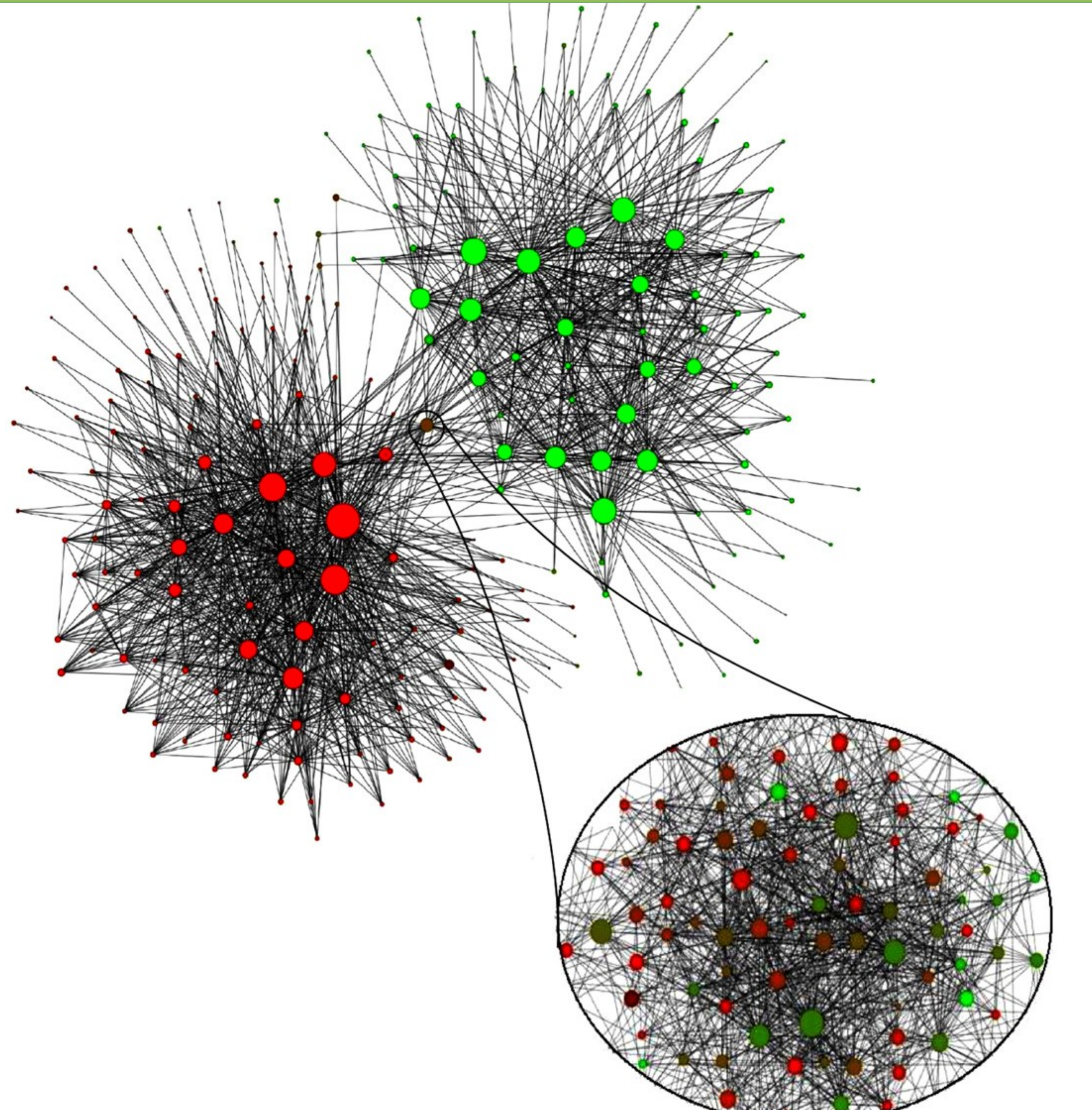
Application: Belgium phone call network

- Link (A – B) : A and B talk frequently on the phone
- Phone calls of ~2 million customers



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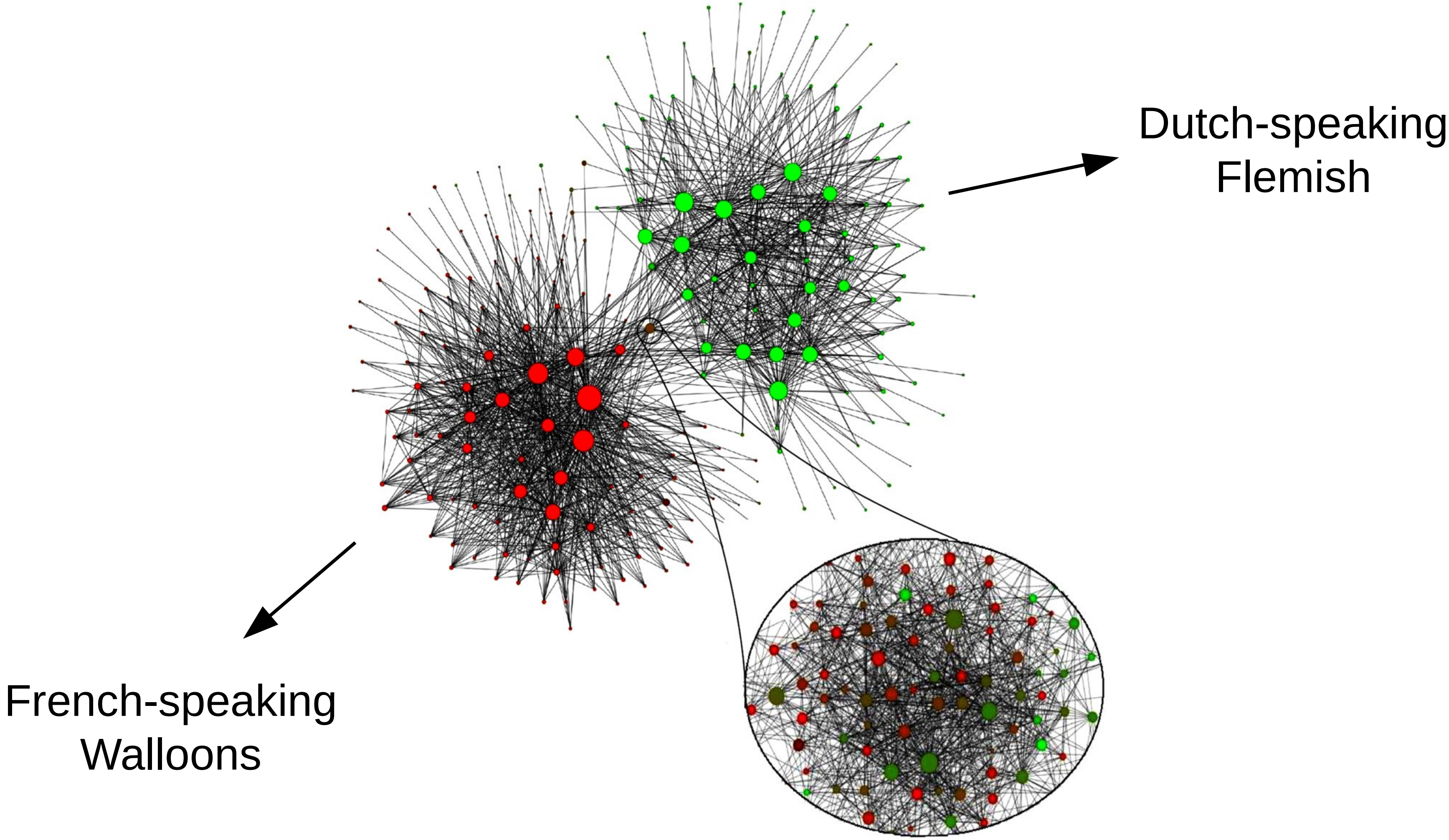


V.D. Blondel et al, *J. Stat. Mech*. P10008 (2008).

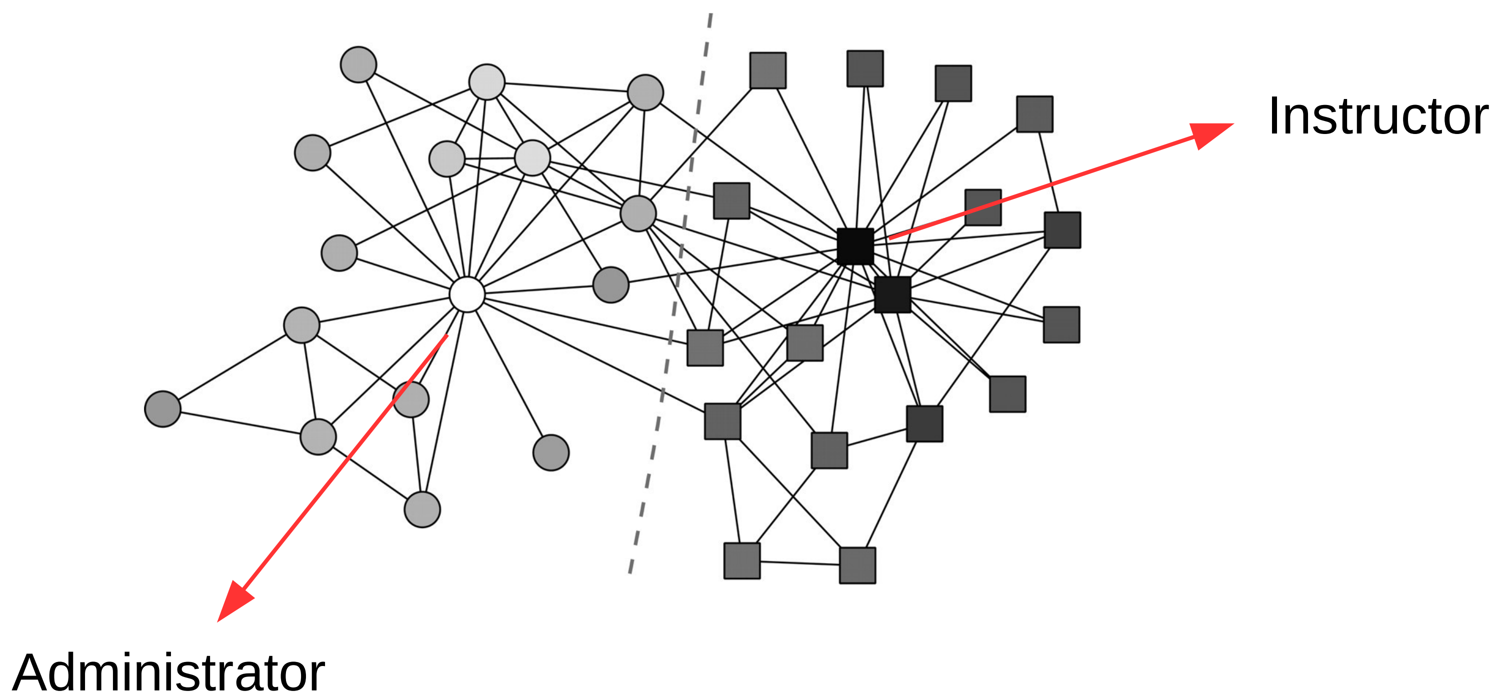
Comparing methods

Comparing methods

- We need a network where we know the true community.
- Option I: Real networks with known ground truth
- Option II: Model networks with built-in communites

Comparing methods

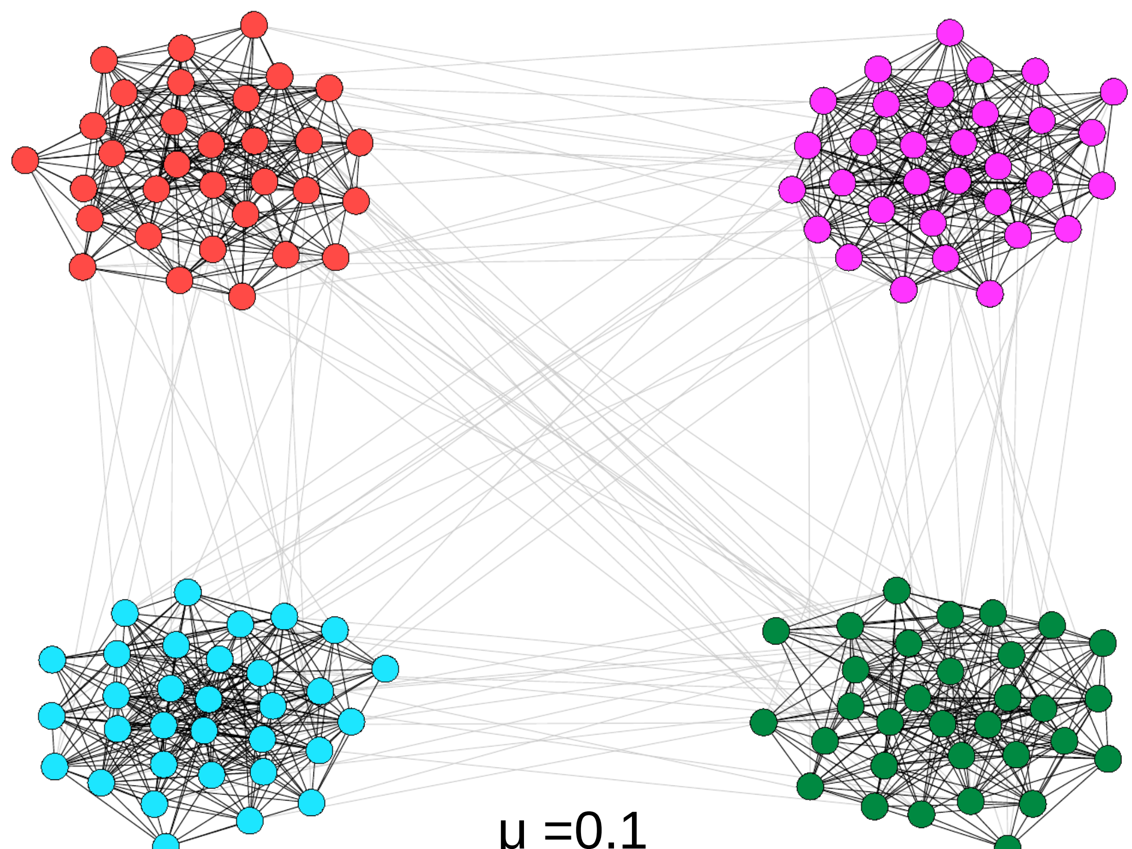
- We need a network where we know the true community.
- Option I: Real networks
- Zachary Karate Club



M. Girvan et al., *PNAS*, 99:12 (2002)
W.W. Zachary, *J. Anthropol. Res.* 33:452-473 (1977).

Artificial benchmarks

- Girvan-Newman benchmark
- N=128 node divided into 4 groups, <k>=16
- p_{in} = prob. that two nodes in the same group are connected
- p_{out} = prob. that two nodes in different groups are connected (not independent)
- μ = fraction of external links = $3p_{out}/(p_{in}+3p_{out})$
- No communities:
  $p_{in}=p_{out}$ or μ =0.75

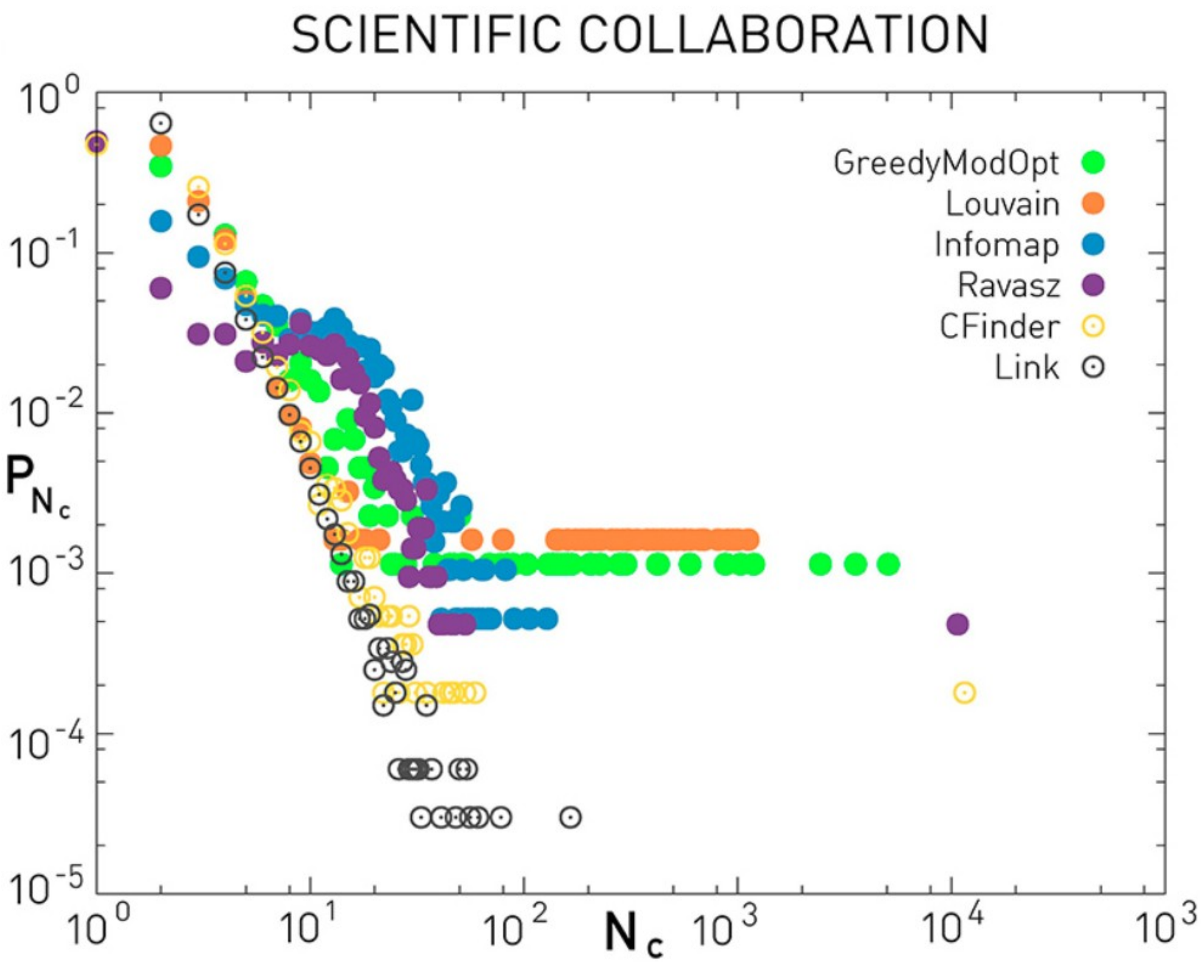


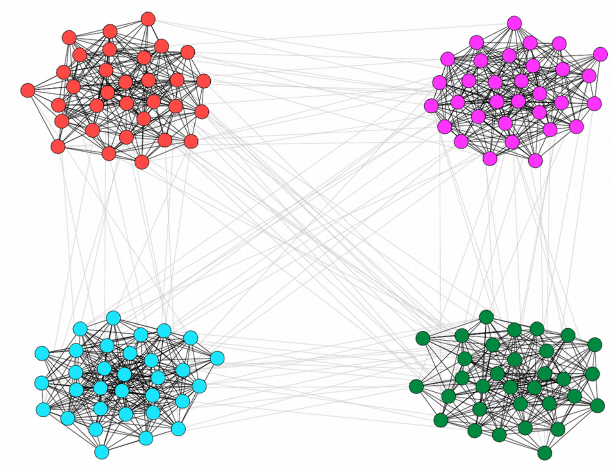
μ =0.1

M. Girvan et al., *PNAS*, 99:12 (2002)

Artificial benchmarks

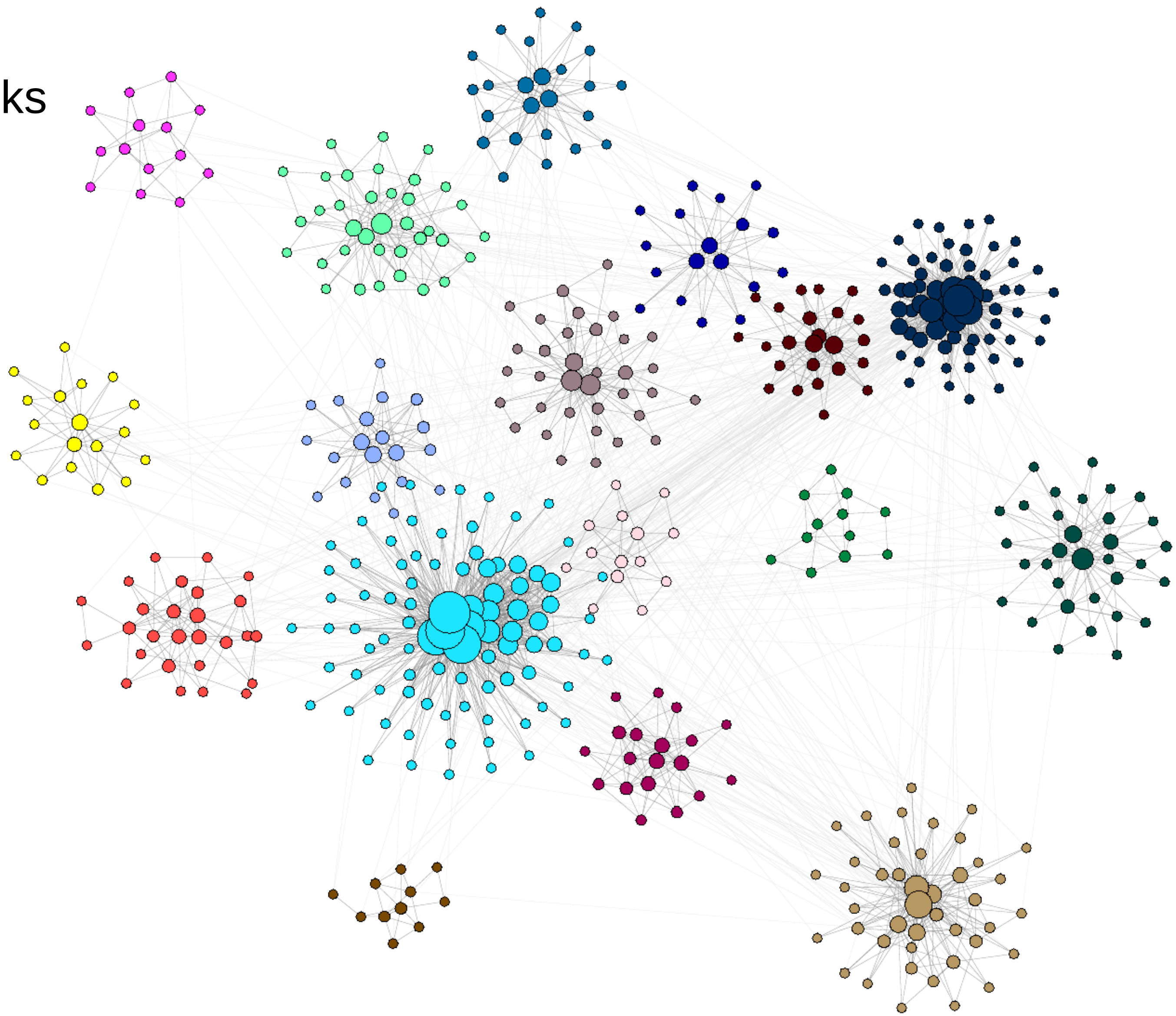
- Is the Girvan-Newman benchmark realistic?
- Community size distribution





Artificial benchmarks

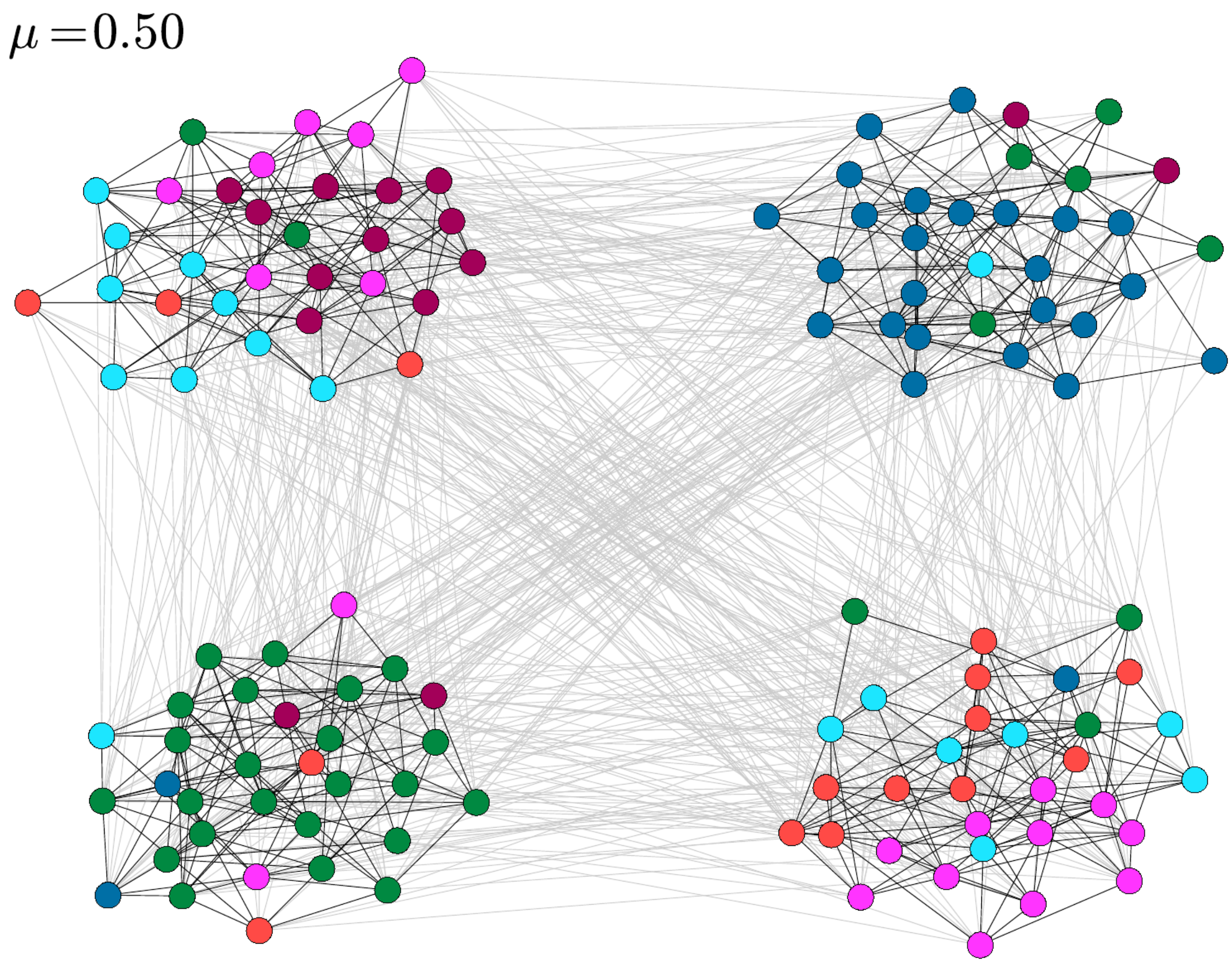
- Lancichinetti–Fortunato–Radicchi (LFR) benchmark
- N nodes, N_c communities
- μ = fraction of external links
- Power-law degree distribution
- Power-law community size distribution



A. Lancichinetti et al., *PRE*, 78:046110 (2008)

Comparing community divisions

- We know what we should get.
- How to systematically compare what we found?
- Again, a lot of possibilities
- Our choice now: Normalized mutual information



Normalized mutual information

Information theory approach: if two partitions are similar, one needs very little information to infer one partition given the other. We can use **the mutual information**

Shannon entropy:

$$H(\{C_1\}) = -\sum_{C_1} p(C_1) \log p(C_1)$$

Measures the amount of information in a string of random variables drawn from p(C_1)

Probability that a randomly chosen node belongs to community C_1

$$p(C_1) = \frac{\text{how many nodes belong to } C_1}{\text{sum over all partitions}} = \frac{N_{C_1}}{\sum_C N_C}$$

L. Danon et al, *JStat.* P09008 (2005)
S. Fortunato, *Phys. Rep.* 486 (2010)

Normalized mutual information

Information theory approach: if two partitions are similar, one needs very little information to infer one partition given the other. We can use **the mutual information**

$$I\left(\{C_1\},\{C_2\}\right) = \sum_{C_1}\sum_{C_2} p\left(C_1, C_2\right) \log \frac{p\left(C_1, C_2\right)}{p\left(C_1\right) p\left(C_2\right)}$$

Joint probability that a randomly chosen node belongs to community C_1 in the first partition and C_2 in the second

$$p\left(C_1, C_2\right) = \frac{\text{how many nodes that are in } C_1 \text{ are also in } C_2}{\text{sum over all possible pairs } C_1 \text{ and } C_2} = \frac{N_{C_1 C_2}}{\sum_{C_1, C_2} N_{C_1 C_2}}$$

Probability that a randomly chosen node belongs to community C_1

$$p(C_1) = \frac{\text{how many nodes belong to } C_1}{\text{sum over all partitions}} = \frac{N_{C_1}}{\sum_C N_C}$$

Normalization by average Shannon entropy:

$$I_n\left(\{C_1\},\{C_2\}\right) = \frac{2I\left(\{C_1\},\{C_2\}\right)}{H\left(\{C_1\}\right) + H\left(\{C_2\}\right)}$$

L. Danon et al, *JStat.* P09008 (2005)
S. Fortunato, *Phys. Rep.* 486 (2010)

Normalized mutual information

- In summary:

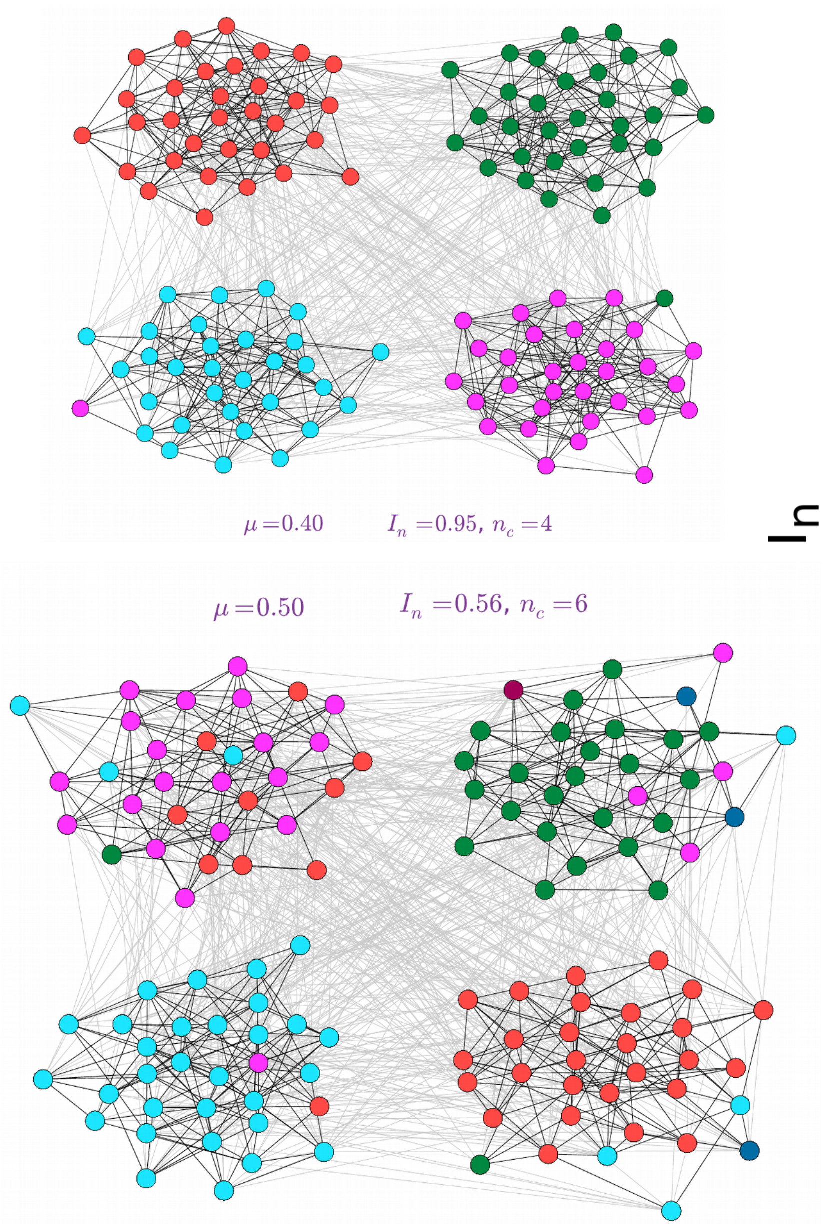
$$I_n\left(\{C_1\},\{C_2\}\right)=\frac{2I\left(\{C_1\},\{C_2\}\right)}{H\left(\{C_1\}\right)+H\left(\{C_2\}\right)}$$

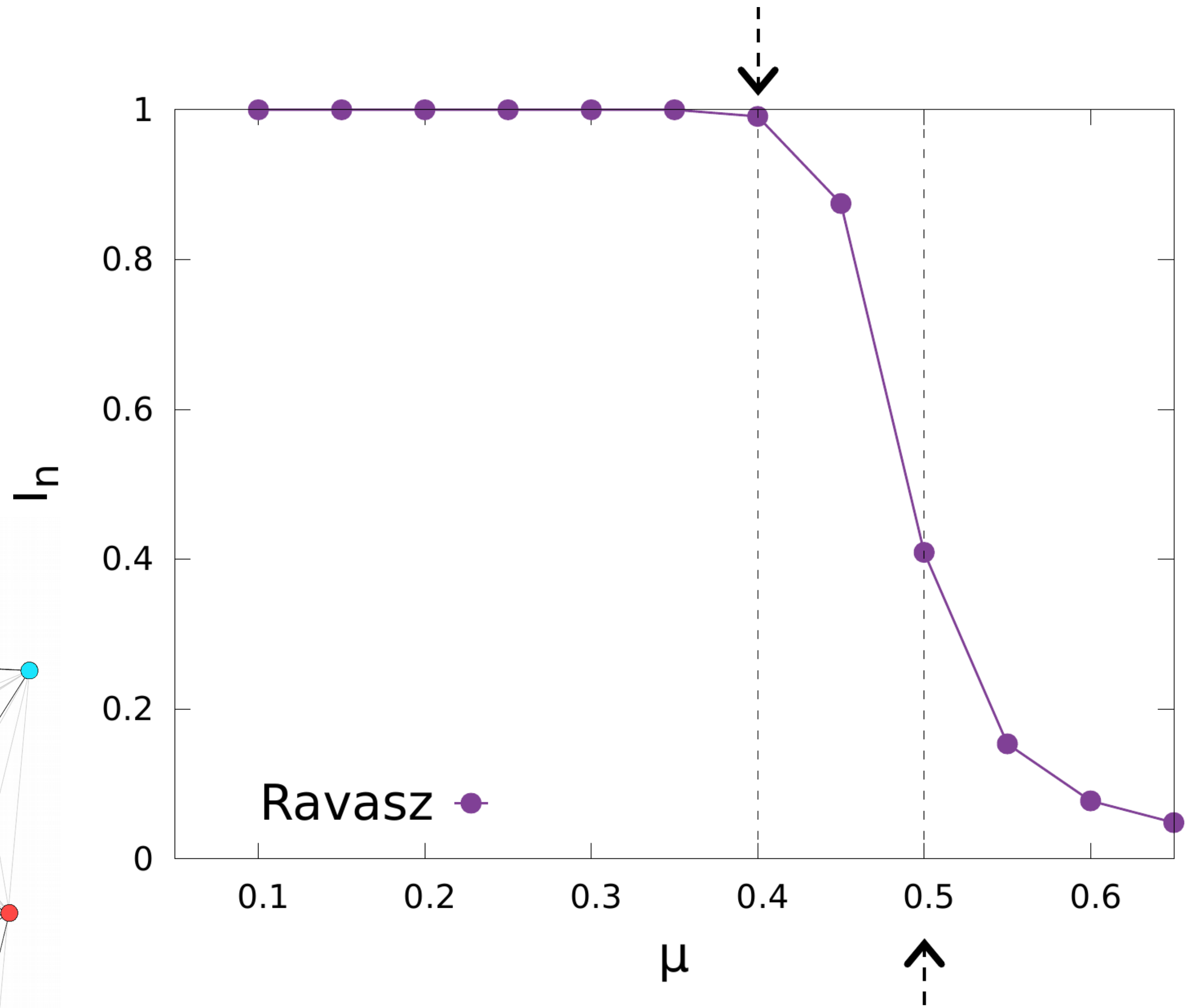
- it quantifies the "amount of information" (in units such as bits) obtained about one random variable, through the other random variable (wiki)
- $I_n = 1 \rightarrow$ same division
- $I_n = 0 \rightarrow$ two divisions independent from each other

L. Danon et al, *JStat.* P09008 (2005)
S. Fortunato, *Phys. Rep.* 486 (2010)

Benchmarks and NMI in action

- NG benchmark, hierarchical clustering





A.-L. Barabási, *Network Science: Communities*.

Benchmarks and NMI in action

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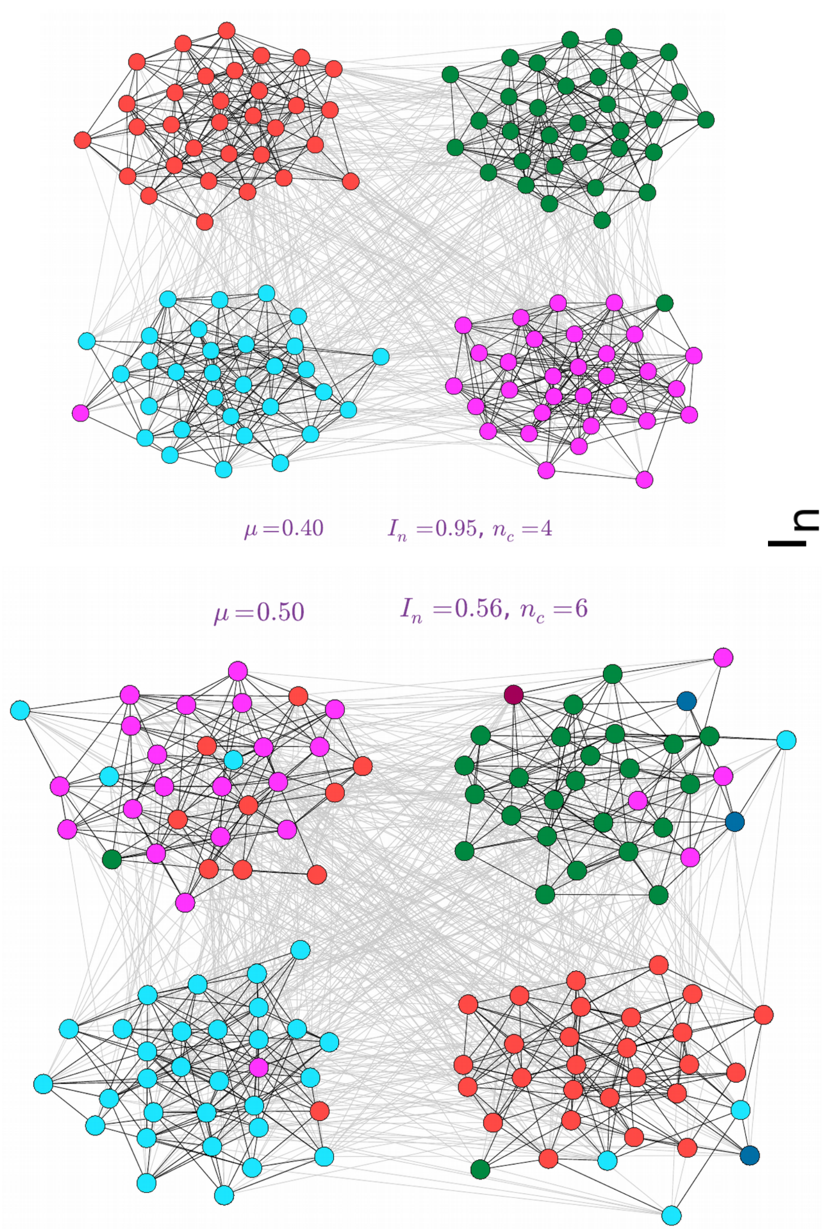
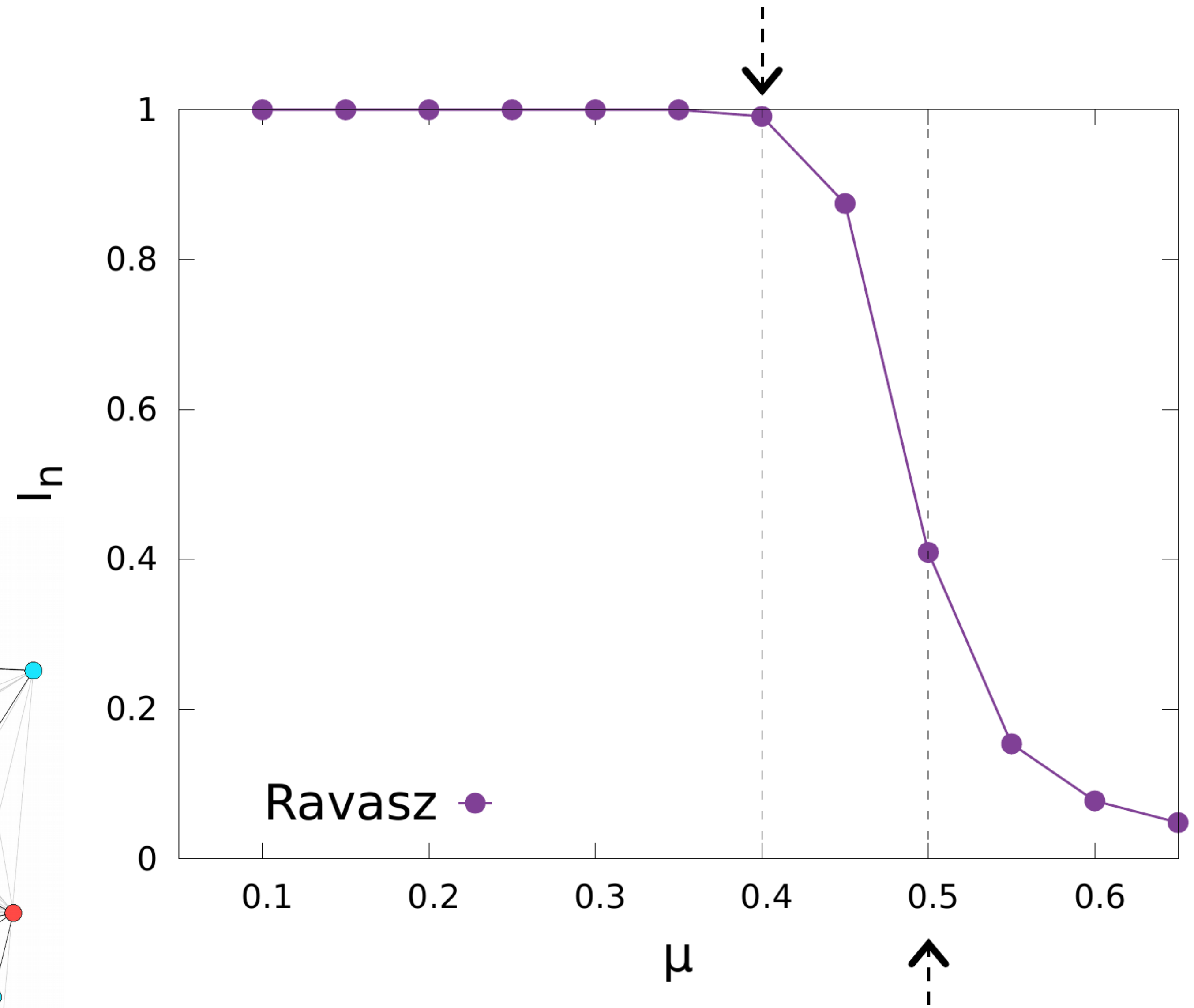
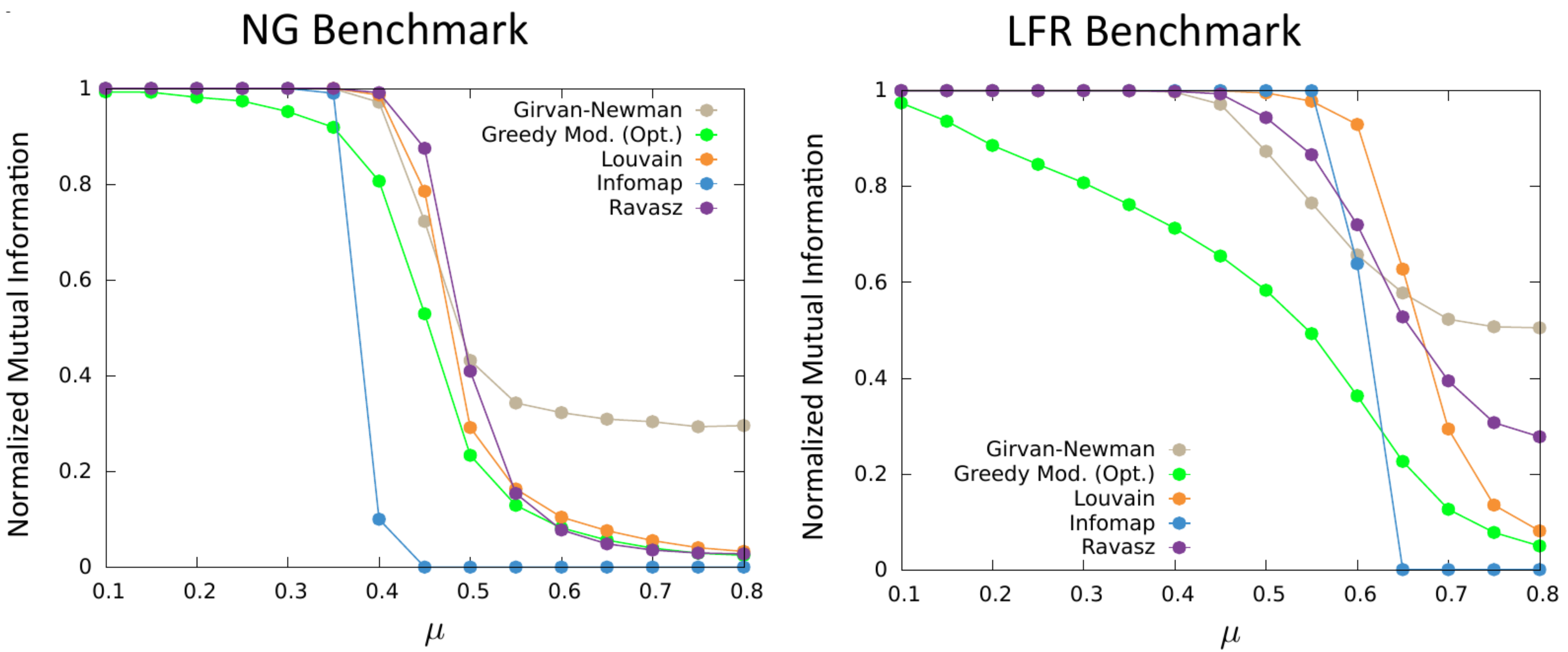




A.-L. Barabási, *Network Science: Communities*.

Benchmarks and NMI in action

- Purple: Hierarchical; Orange: Louvain; Gray: Betweenness



LFR parameters: N = 1000; degree exp. = 2; max degree = 50; comm. size exponent = 1, min comm. size = 20, max = 100

A.-L. Barabási, *Network Science: Communities*.

Short list of other methods

- Many other methods to find communities:
  - Local: instead of finding global division, find the community a given node belongs to
  - Spectral: based on spectrum of graph Laplacian
  - Dynamical: Potts-model, oscillators, random walks
  - Stochastic block models: find best fit using maximum likelihood fit of benchmark-like model → mathematically principled results

How to choose method?

- What is the best method?

  → No clear answer.

- Better question: What is the method that fits my needs?
  - Network features: Size? Directed? Weighted? Bipartite?
  - What do we expect to find? Overlapping communities? Size of the groups?

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How to choose method?

	Name	Overlap	Dir	Weight	Dyn	NoPar	MDim	Incr	Multip	Complexity	BESn	BESm	Year	Ref
Feature Distance	Evolutionary*				✓			✓		$\mathcal{O}(n^2)$	5k	?	2006	[21]
	MSN-BD			✓					✓	$\mathcal{O}(n^2ck)$	6k	3M	2006	[22]
	SocDim	✓		✓			✓			$\mathcal{O}(n^2 \log n)*$	80k	6M	2009	[23]
	PMM			✓			✓			$\mathcal{O}(mn^2)$	15k	27M	2009	[24]
	MRGC		✓		✓		✓		✓	$\mathcal{O}(mD)$	40k	?	2007	[25]
	Infinite Relational					✓	✓			$\mathcal{O}(n^{2c}D)$	160	?	2006	[26]
	Find-Tribes				✓				✓	$\mathcal{O}(mnK^2)$	26k	100k	2007	[27]
	AutoPart		✓			✓		✓		$\mathcal{O}(mk^2)$	75k	500k	2004	[28]
	Timefall				✓	✓			✓	$\mathcal{O}(mk)$	7.5M	53M	2008	[29]
	Context-specific Cluster Tree					✓			✓	$\mathcal{O}(mk)$	37k	367k	2008	[30]
IntDensity	Modularity	✓	✓	✓			✓	✓	✓	$\mathcal{O}(mk \log n)$	118M	1B	2004	[18]
	MetaFac				✓		✓			$\mathcal{O}(mnD)$	?	2M	2009	[31]
	Variational Bayes		✓			✓				$\mathcal{O}(mk)$	115	613	2008	[32]
	$LA \to IS^2*$	✓	✓							$\mathcal{O}(mk+n)$	16k	?	2005	[33]
	Local Density		✓			✓		✓		$\mathcal{O}(nK \log n)$	108k	330k	2005	[34]
Bridge	Edge Betweenness		✓	✓						$\mathcal{O}(m^2n)$	271	1k	2002	[4]
	CONGO*	✓		✓						$\mathcal{O}(n \log n)$	30k	116k	2008	[35]
	L-Shell	✓						✓		$\mathcal{O}(n^3)$	77	254	2005	[36]
	Internal-External Degree	✓								$\mathcal{O}(n^2 \log n)$	775k	4.7M	2009	[37]
Diffusion	Label Propagation			✓		✓		✓		$\mathcal{O}(m+n)$	374k	30M	2007	[38]
	Node Colouring				✓				✓	$\mathcal{O}(ntk^2)$	2k	?	2007	[39]
	Kirchhoff	✓		✓						$\mathcal{O}(m+n)$	115	613	2004	[40]
	Communication Dynamic	✓	✓		✓			✓		$\mathcal{O}(mnt)$	160k	530k	2008	[41]
	GuruMine		✓		✓					$\mathcal{O}(TAn^2)$	217k	212k	2008	[9]
	DegreeDiscountIC		✓							$\mathcal{O}(k \log n + m)$	37k	230k	2009	[42]
	MMSB	✓	✓							$\mathcal{O}(nk)$	871	2k	2007	[43]
Close	Walktrap			✓						$\mathcal{O}(mn^2)$	160k	1.8M	2006	[44]
	DOCS	✓								?	325k	1M	2009	[45]
	Infomap		✓	✓		✓				$\mathcal{O}(m \log^2 n)$	6k	6M	2008	[46]
Structure	K-Clique	✓								$\mathcal{O}(m^{\frac{\ln m}{10}})$	20k	127k	2005	[3]
	S-Plexes Enumeration	✓								$\mathcal{O}(kmn)$	?	?	2009	[47]
	Bi-Clique	✓							✓	$\mathcal{O}(m^2)$	200k	500k	2008	[48]
	EAGLE	✓	✓	✓						$\mathcal{O}(3^{\frac{n}{3}})$	16k	31k	2009	[49]
Link	Link modularity	✓		✓					✓	$\mathcal{O}(2mk \log n)$	20k	127k	2009	[50]
	HLC*	✓		✓					✓	$\mathcal{O}(n\bar{K}^2)$	885k	5.5M	2010	[51]
	Link Maximum Likelihood	✓								$\mathcal{O}(mk)$	4.8M	42M	2011	[52]
NoD	Hybrid*	✓	✓	✓		✓				$\mathcal{O}(nkK)$	325k	1.5M	2010	[53]
	Multi-relational Regression			✓			✓			?	?	?	2005	[54]
	Hierarchical Bayes									$\mathcal{O}(n^2)$	1k	4k	2008	[55]
	Expectation Maximization		✓							?	112	?	2007	[7]

M. Coscia et al, *Statistical Analysis and Data Mining,* 4(5):512546 (2011)

Literature

- Where to start reading?

1) Newman, M. E., Communities, modules and large-scale structure in networks. Nature Physics, 8(1):25-31 (2012).
→ Short, big picture

2) Fortunato, S., Community detection in graphs. Physics reports, 486(3):75-174 (2010).
→ >100 pages, complete at the time, good for looking up methods

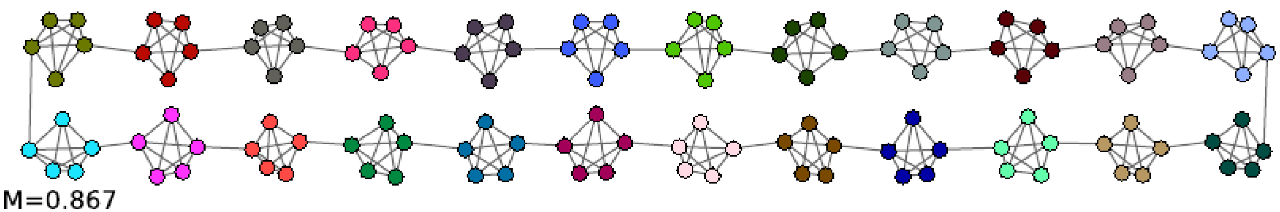
3) Coscia, M., Giannotti, F., & Pedreschi, D., A classification for community discovery methods in complex networks. Statistical Analysis and Data Mining, 4(5):512-546 (2011).
→ shorter, compares a lot of methods

4) A.-L. Barabási, Network Science, Chapter 9
http://barabasi.com/networksciencebook/
Appears in print in May.
→ Lot of figures of the lecture are from here. Easy to read, tells a detailed story, but does not cover everything.

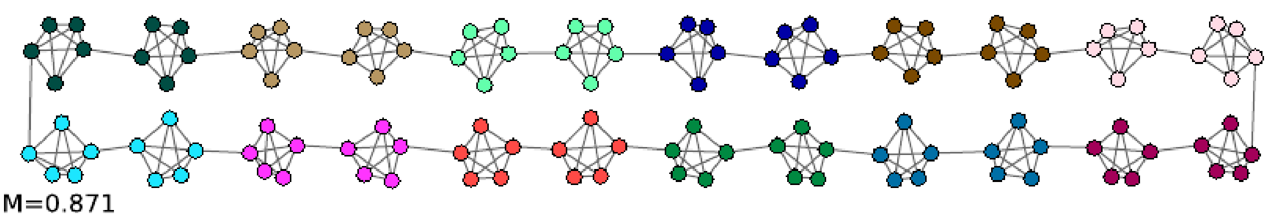
Extra time:
Problems with modularity

Is the maximum unique?

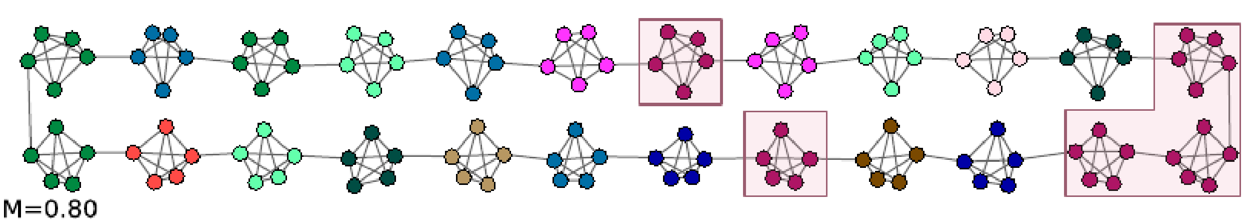
- Should we merge two communities?
- Intuitive:

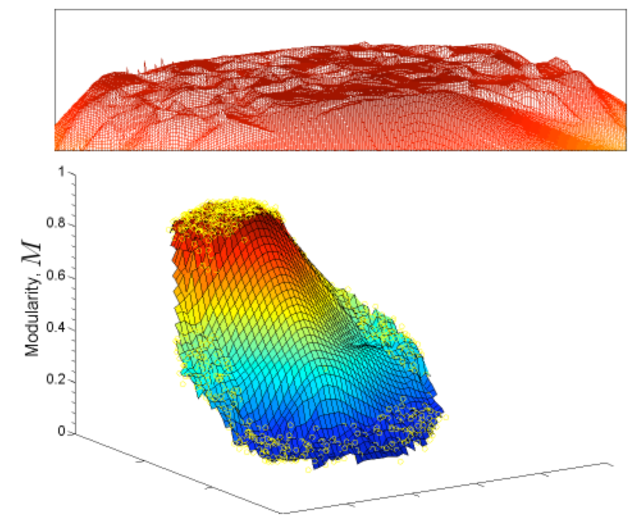


- Global maximum:



- Random:



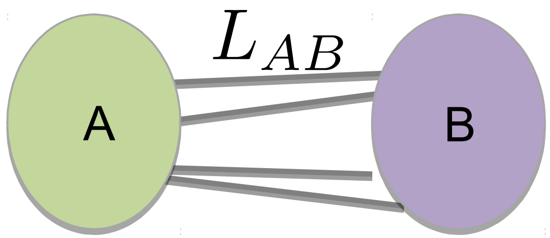


Good et al., *PRE*, 81:046106 (2010)

A.-L. Barabási, *Network Science: Communities*.

Resolution limit

- Should we merge two communities?



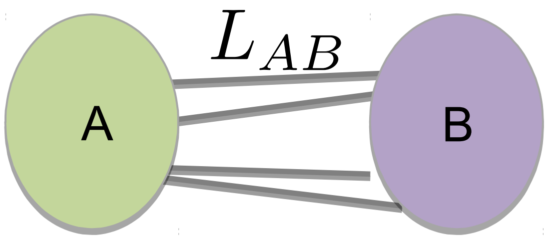
$$\Delta M_A = \frac{1}{2L}\sum_{i,j\in A} A_{ij} - \frac{k_i k_j}{2L} = \frac{2k_A^{\mathrm{int}}}{2L} - \frac{1}{2L}\sum_{i,j\in A}\frac{k_i k_j}{2L}$$

$$\begin{aligned}
\Delta M_{AB} &= -M_A - M_B + M_{AB} = \\
&= -\left(\frac{k_A^{\mathrm{int}}}{L} - \sum_{i,j\in A}\frac{k_i k_j}{(2L)^2})\right) - \left(\frac{k_B^{\mathrm{int}}}{L} - \sum_{i,j\in B}\frac{k_i k_j}{(2L)^2})\right) + + \\
&+ \left(\frac{k_A^{\mathrm{int}} + k_A^{\mathrm{int}} + L_{AB}}{L} - \sum_{i,j\in A}\frac{k_i k_j}{(2L)^2} - \sum_{i,j\in B}\frac{k_i k_j}{(2L)^2} - 2\sum_{i\in A, j\in B}\frac{k_i k_j}{(2L)^2}\right) = \\
&= \frac{L_{AB}}{L} - \frac{k_A k_B}{2L^2}
\end{aligned}$$

S. Fortunato. *PNAS*, 104(1):36-41 (2006)
A.-L. Barabási, *Network Science: Communities*.

Resolution limit

- Should we merge two communities?



$$\Delta M_{AB} = \frac{L_{AB}}{L} - \frac{k_A k_B}{2L^2},$$

k_A and k_B total degree in A and B

If $\frac{k_A k_B}{2L} < 1$ and $L_{AB} \geq 1$ ⇒ $\Delta M_{AB} > 0$ We merge A and B to maximize modularity.

Assuming $k_A \sim k_B = k$ ⇒ $k \leq \sqrt{2L}$

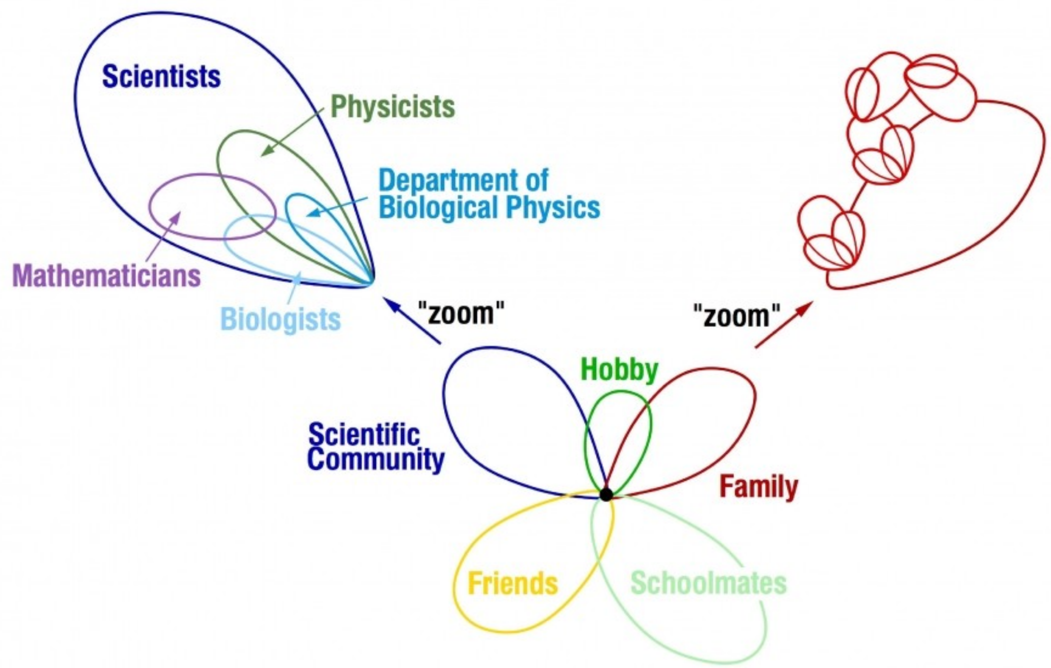
Modularity has a resolution limit, as it cannot detect communities smaller than this size.

S. Fortunato. *PNAS*, 104(1):36-41 (2006)

A.-L. Barabási, *Network Science: Communities*.

Even more time:
Link communities

Link communities



Nodes tend to belong to multiple communities

Links tend to be specific, capturing the nature of the relationship between two nodes.

Social networks, a link may indicate:

- they are in the same family; they work together; they share a hobby.

Biological networks:
each interaction of a protein is responsible for a different function, uniquely defining the protein's role in the cell

Define a hierarchical algorithm based on similarity of links

Ahn, Bragow and Lehmann, *Nature* 466 (2010).
A.-L. Barabási, *Network Science: Communities*.
Palla, et al. Nature, 435:814, 2005

Link communities

1. **Define link similarity**

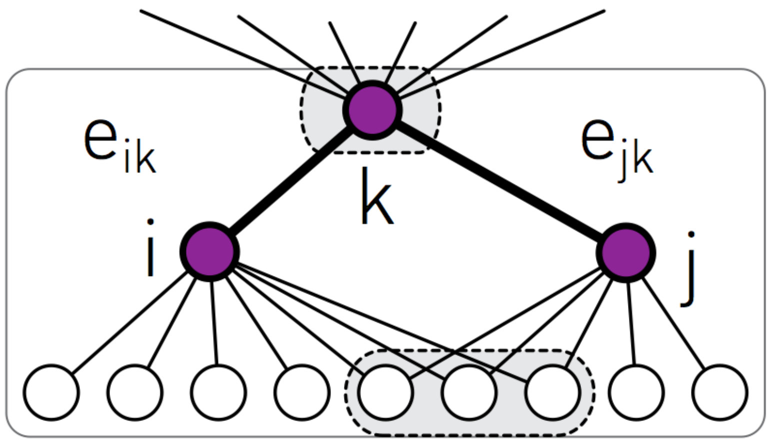
$$S(e_{ik}, e_{jk}) = \frac{|n_+(i) \cap n_+(j)|}{|n_+(i) \cup n_+(j)|}$$

$n_+(i)$: the list of the neighbors of node i, including itself.
S measures the relative number of common neighbors i and j have.

$$\left| n_+(i) \cap n_+(j) \right| = 4$$
$$\left| n_+(i) \cup n_+(j) \right| = 12$$

$$S\left(e_{ik}, e_{jk}\right) = \frac{1}{3}$$

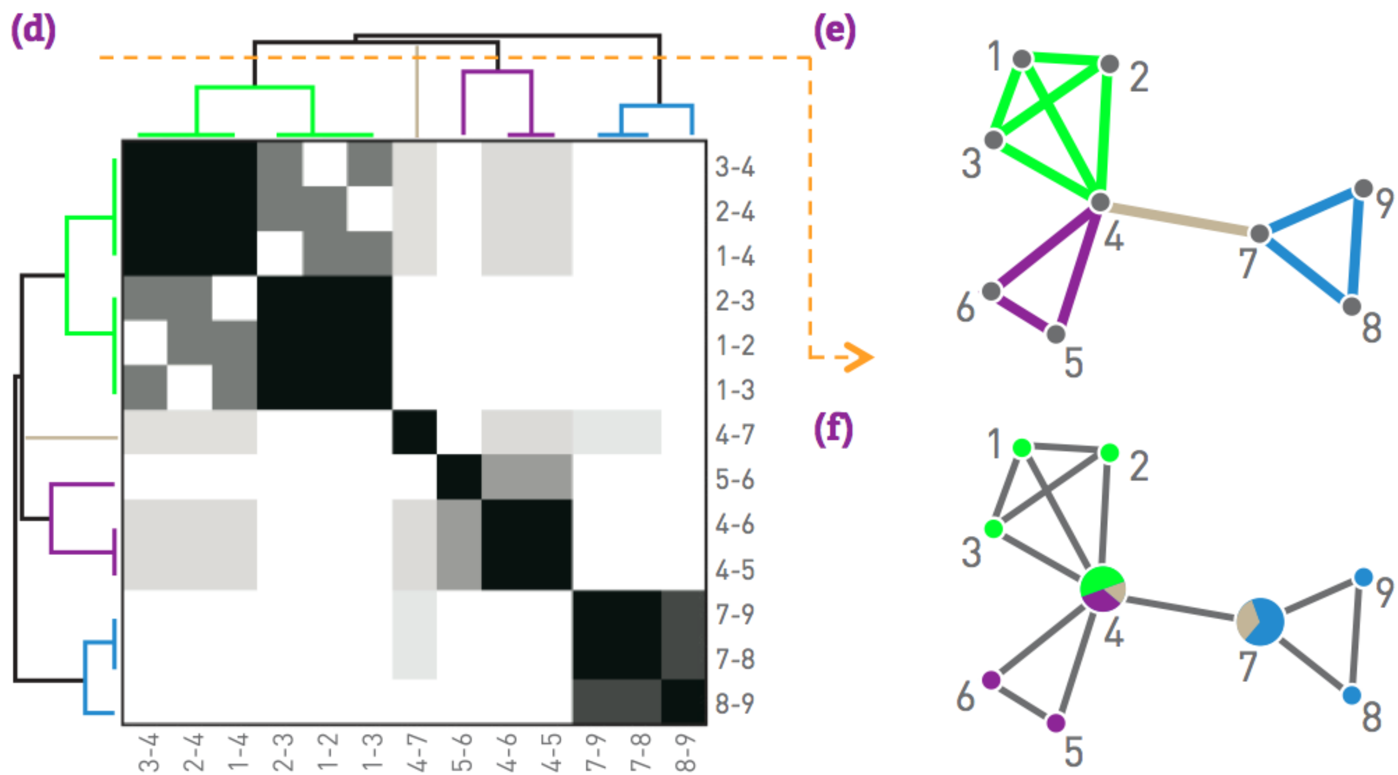
(a)



Ahn, Bragow and Lehmann, *Nature* 466 (2010).
A.-L. Barabási, *Network Science: Communities*.

Link communities

2. **Apply hierarchical clustering (agglomerative, single linkage)**



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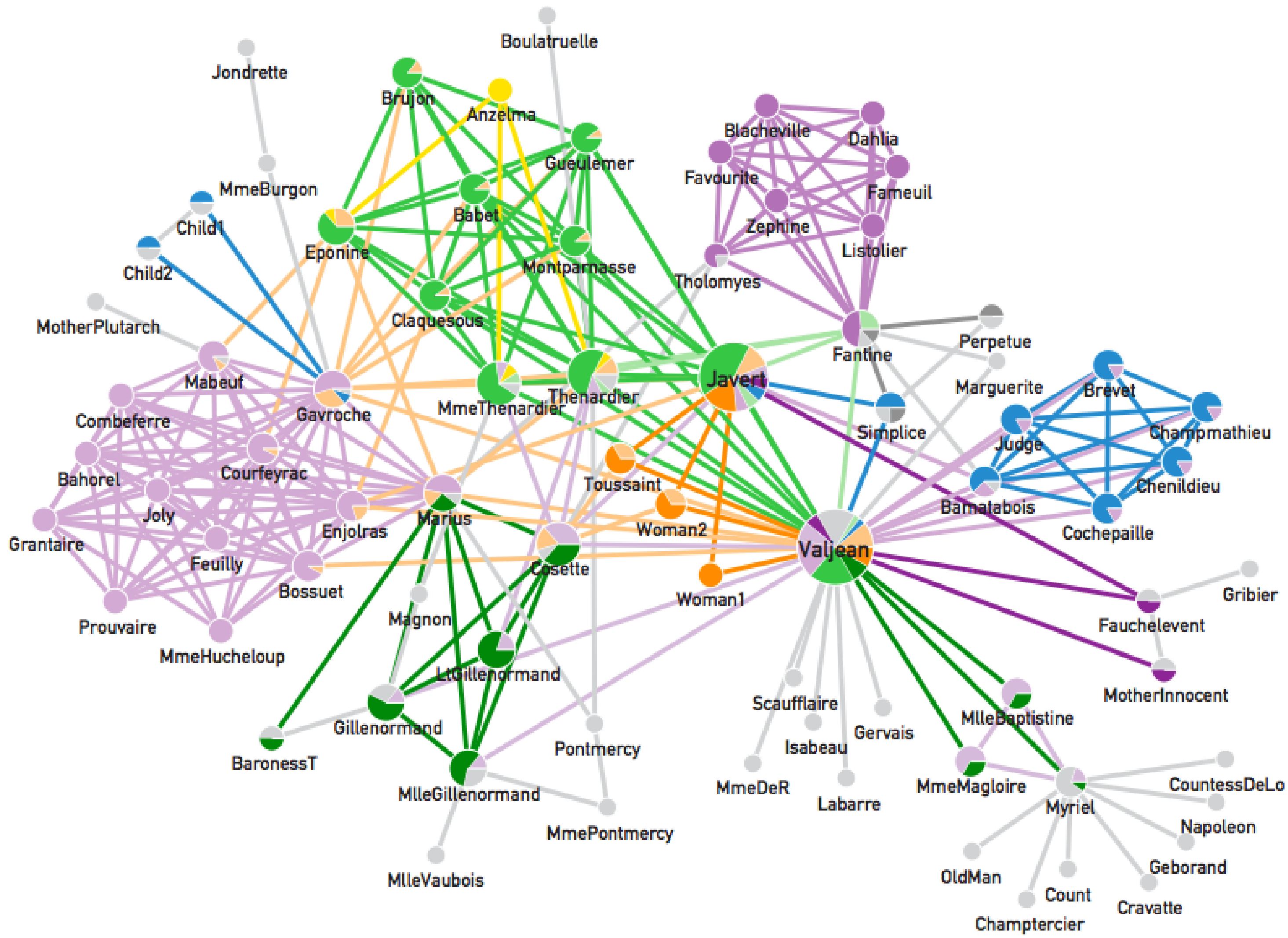
Link communities



The network of characters in Victor Hugo's 1862 novel *Les Miserables*. Two characters are connected if they interact directly with each other in the story. The link colors indicate the clusters, grey nodes corresponding to single-link clusters. Each node is depicted as a pie-chart, illustrating its membership in multiple communities. Not surprisingly, the main character, Jean Valjean, has the most diverse community membership

Ahn, Bragow and Lehmann, *Nature* 466 (2010).
A.-L. Barabási, *Network Science: Communities*.