ECS 253 / MAE 253, Network Theory and Applications
Spring 2023
Common Problem Set # 1, Due April 19

Problem 1: Power Law Degree Distributions

Consider the power law distribution $p(k) = Ak^{-\gamma}$, with support (i.e., defined from) $k = 1$ to $k \to \infty$. In the steps below, you can either treat the k's using a continuum approximation (as we did in class) or you can treat the k's as discrete. The continuum approximation:

$$\sum_{k=1}^{\infty} p_k \approx \int_{k=1}^{\infty} p_k dk$$

The exact treatment:

$$\sum_{k=1}^{\infty} p(k) = \sum_{k=1}^{\infty} A'k^{-\gamma} = A'\left[1 + \frac{1}{2^\gamma} + \frac{1}{3^\gamma} + \frac{1}{4^\gamma} + \cdots\right].$$

The sum in the brackets is known as the Riemann Zeta Function, $RZ(\gamma)$. The value of $RZ(\gamma)$, for many values of γ can be found in standard references (*e.g.*, Mathworld, Wikipedia, etc).

a) Show that we must have $\gamma > 1$ for this to be a properly defined probability distribution function (pdf). Recall a pdf must have two properties: 1) $p(k) \geq 0$ for all k, and 2) it must be normalized.

b) Solve for the normalization constant A.

c) Show that if $1 < \gamma \leq 2$, the average value $\langle k \rangle$ diverges.

d) Show that if $2 < \gamma \leq 3$, the average is finite, but the variance, σ^2, diverges. The best way to do this is to realize $\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2$, explicitly:

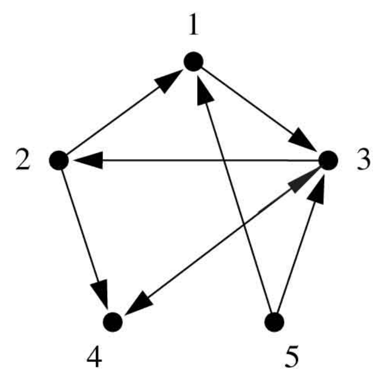
$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N} (k_i - \langle k \rangle)^2 = \frac{1}{N}\left[\sum_i (k_i)^2 - 2\sum_i k_i \langle k \rangle + \sum_i \langle k \rangle^2\right] = \left\langle k^2 \right\rangle - \langle k \rangle^2.$$

e) Plot $p(k) = Ak^{-\gamma}$, for $k = 1$ to $k = 100,000$ for $\gamma = 3$, and properly normalize. Use matlab, R, or pen and paper, etc (and make sure to label axes clearly with values).

(A note on "properly normalizing": The continuum approach (as shown in class) leads to $A = \gamma - 1$, meaning for $\gamma = 3$ then $p(k) = 2k^{-3}$ (i.e. $p(1) > 1$!). Thus the continuum approximation actually leads to the wrong normalization constant! To properly normalize you either need the Riemann Zeta func approach or you have to explicitly calculate (e.g., by writing a little computer code) $\sum_{k=1}^{m} A' k^{-3} = 1$ where m is a big number, say $m = 10^6$, and solving for the correct normalization constant $A' \neq \gamma - 1$.)

f) In a finite network with N nodes, what is the largest possible value of degree, $k_{\max}$, that can ever be observed? So can we ever have $\langle k \rangle \to \infty$ in a finite network?

Problem 2: Adjacency matrix



a) Consider the simple network shown above and write down its the adjacency matrix.

b) Consider a random walk on this network. What is the steady-state probability of finding the walker on each node?

c) What would be the steady-state probability of finding the walker on each node if the edges were instead *undirected*?

Problem 3: Rate equations: Network growth with uniform attachment

Consider a variant of the BA model that does not feature preferential attachment. We start with a single node at time $t = 1$. In each subsequent discrete time step, a new node is added with $m = 1$ links to existing nodes. The probability that a link arriving at time step $t + 1$ connects to any existing node i is uniformly distributions and independent of i:

$$\pi_i = \frac{1}{t}. \tag{1}$$

Let $n_{k,t}$ denote the expected number of nodes of degree k at time t. For the steps below, proceed as in lecture.

a) Write the rate equation for $n_{k,t+1}$ in terms of the $n_{j,t}$'s. (Note you will need to equations, one for $k = 1$ and one for $k > 1$.)

b) Converting from expected number of nodes to probabilities, $p_{k,t} = n_{k,t}/n_t$, rewrite the equations in part (a) in terms of the probabilities.

c) Assume steady-state, that $p_{k,t} = p_k$, and solve the recurrence relation to obtain p_k in terms of p_{k-1}.

d) Starting by solving for p_1 and recursing, derive the expression for the stationary degree distribution p_k.