ECS 253 / MAE 253, Network Theory and Applications
Spring 2023
Advance Problem Set # 3b, Due May 15

Problem 1: Small world networks

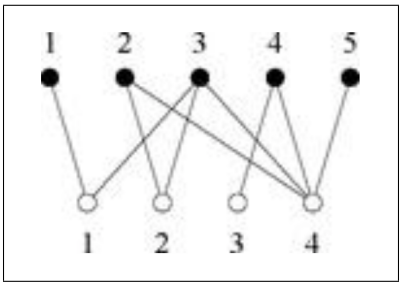
Consider the original Watts-Strogatz small world network, where each node starts on a regular ring with degree c, and each edge is rewired with probability p. When an edge is rewired, both new end points are chosen at random. One of the difficulties is that a node can become disconnected from the rest of the network by the rewiring process. For instance, a single vertex can become disconnected if all of its incident edges are rewired and in addition it gains no shortcut edges.

(a) Show that the probability of this happening to any given vertex is $[pe^{-p}]^c$

(b) Hence, how large must the network be before we expect that one vertex will be disconnected if $c = 6$ and $p = 0.01$?

Problem 2: Bipartite networks

Consider the bipartite network show here:



(a) Write down the incidence matrix of the network. (Note, Wikipedia and Wolfram MathWorld both give good definitions and examples of incidence matrices.)

(b) Write down the projection matrix for the projection of the network onto its black vertices.

Problem 3: User versus system optimal

Consider the following variation of Pigou's congestion example discussed. Two roads connect source s and target t; x and y fraction of cars take Road 1 and Road, respectively; therefore $x + y = 1$. The roads have different travel times that also depend on the traffic:

- Road 1 is very short, but gets easily congested; travel time in hours is given by provided by $T_1(x) = x$.
- Road 2 is the longer and it takes different amount of traffic to get congested; the travel time is $T_2(y) = ay + b$, where $a > 0$ and $b > 0$.

(a) If all drivers are selfish, each chooses the route that yields the shortest drive. Thus, in equilibrium, all routes take the same time T. What is the distribution of cars in this case? What is the average travel time? At what parameter values does all the traffic travel on Road 1?

(b) The system optimum corresponds to the shortest average travel time. What is the distribution of cars in this case? What is the average travel time? At what parameter values does all the traffic travel on Road 1?

(c) Show that the selfish average time can never be equal to the system optimal if $b < 2$.