Some network flow problems in urban road networks

Michael Zhang
Civil and Environmental Engineering
University of California Davis

Outline of Lecture

- Transportation modes, and some basic statistics
- Characteristics of transportation networks
- Flows and costs
- Distribution of flows
  - Behavioral assumptions
  - Mathematical formulation and solution
  - Applications

Vehicle Miles of Travel: by mode

(U.S., 1997, Pocket Guide to Transp.)

Mode	Vehicle-miles (millions)	
Air Carriers	4,911	0.3%
General Aviation	3,877	0.2%
Passenger Cars	1,502,000	88%
Trucks		11%
Single Unit	66,800	4%
Combination	124,500	7%
Amtrak(RAIL)	288	0.0%

Passenger miles by mode

(U.S., 1997, Pocket Guide to Transp.)

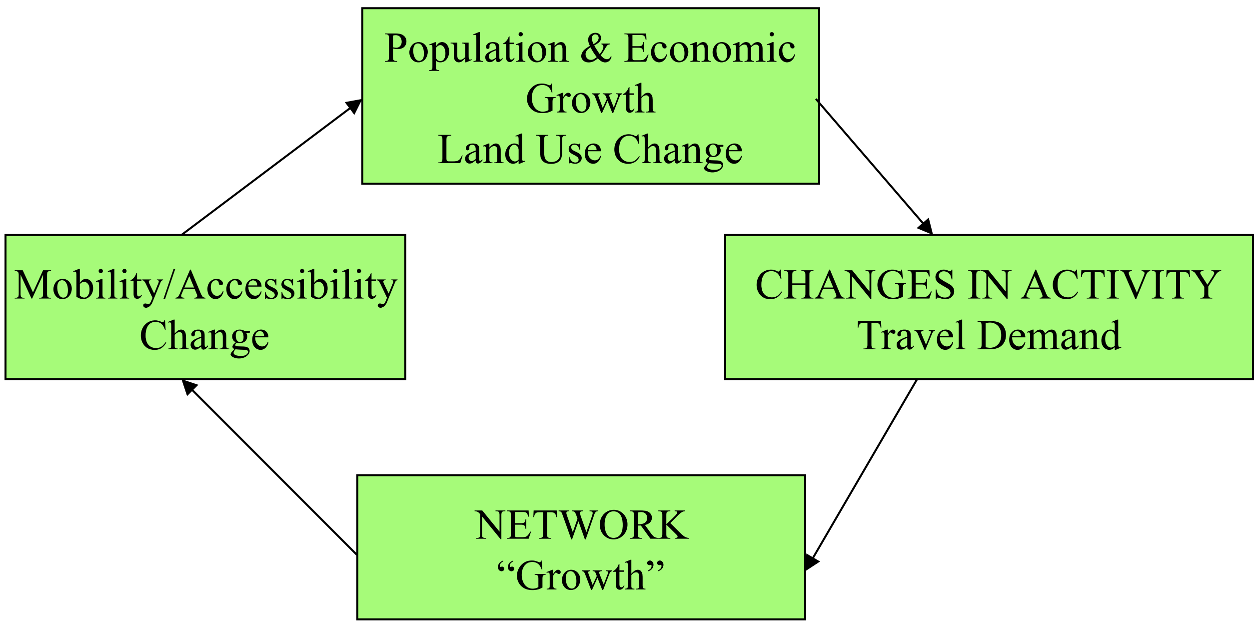
Mode	Passenger-miles (millions)	% SHARE
Air Carriers	450,600	9.75%
General Aviation	12,500	0.27%
Passenger Cars	2,388,000	51.67%
Other vehicles	1,843,100	34.56%
Buses	144,900	3.14%
Rail	26,339	0.56%
Other	1,627	0.04%

Fatalities by mode (1997, US)

Mode	# of fatalities	# per million pas.-mile		
Air	631	0. 001363		
Highway	42013	0. 00993		
Railroad	602	0. 022856		
Transit	275	0. 001898		
Waterborne	959	N/A		

How to "grow" a transportation system:

pop. & economic growth, land use and demand/supply balance



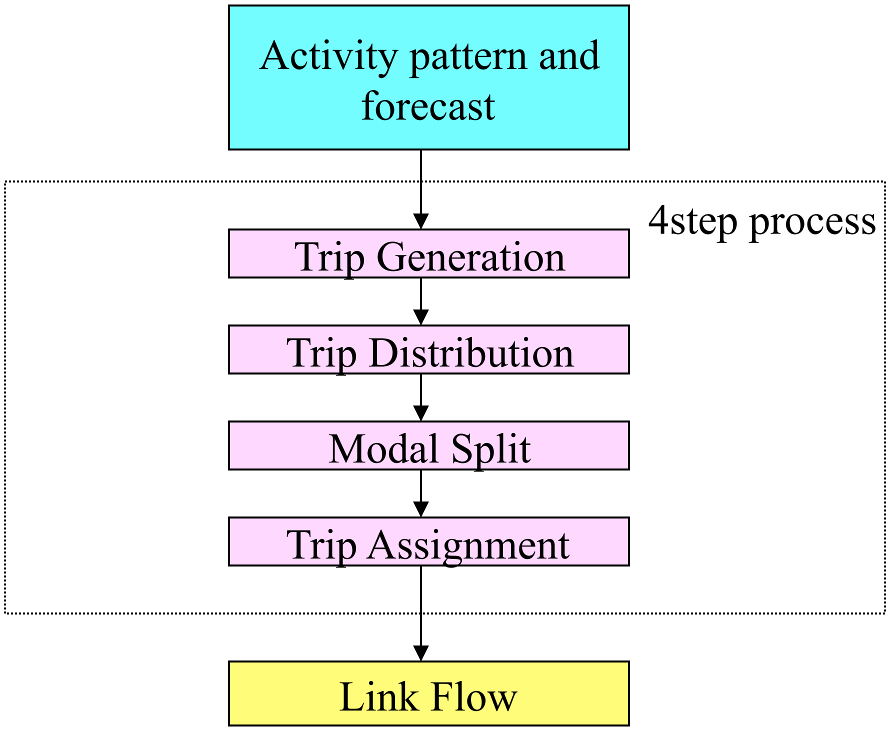
An example: Beijing, China

Population: 5.6 million (1986) -> 10.8 million (2000)
GDP: ~9-10% annual growth

Changes in land use

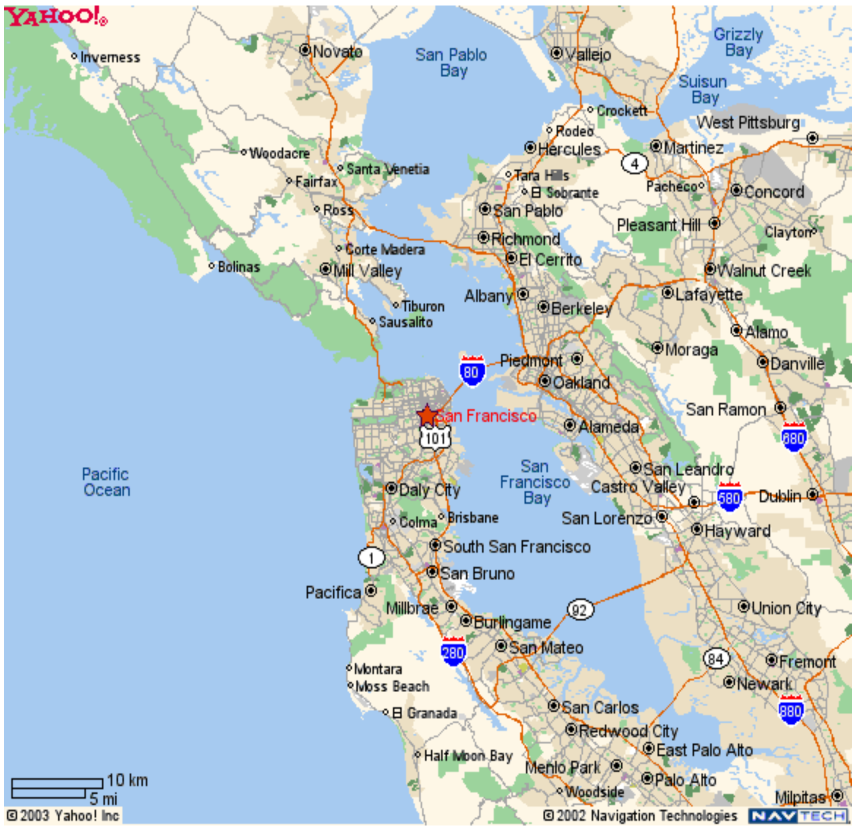
Changes in the highway network

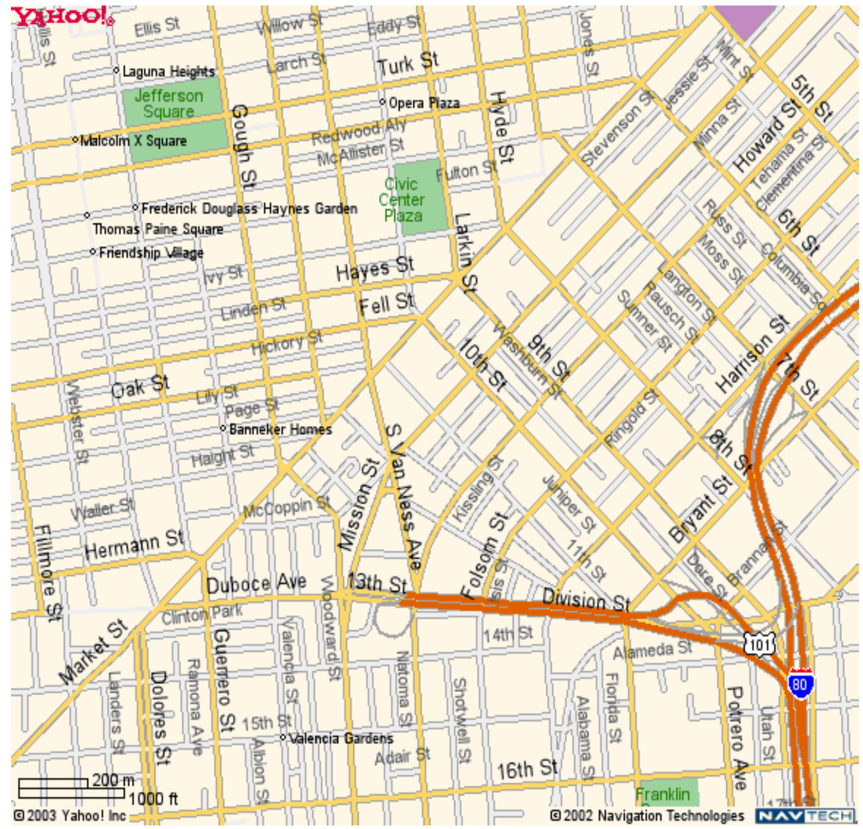
The four step planning process



NEED FEEDBACK

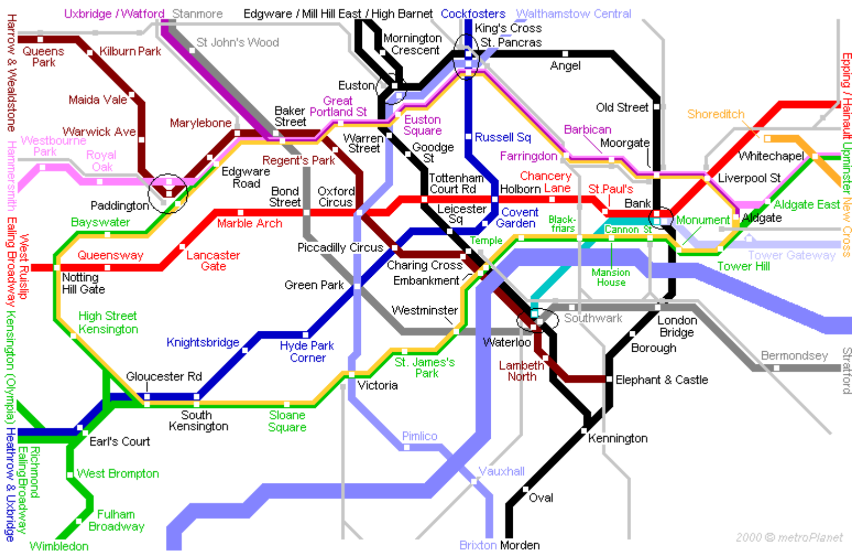
EXAMPLE 1: HIGHWAY TRANSPORTATION





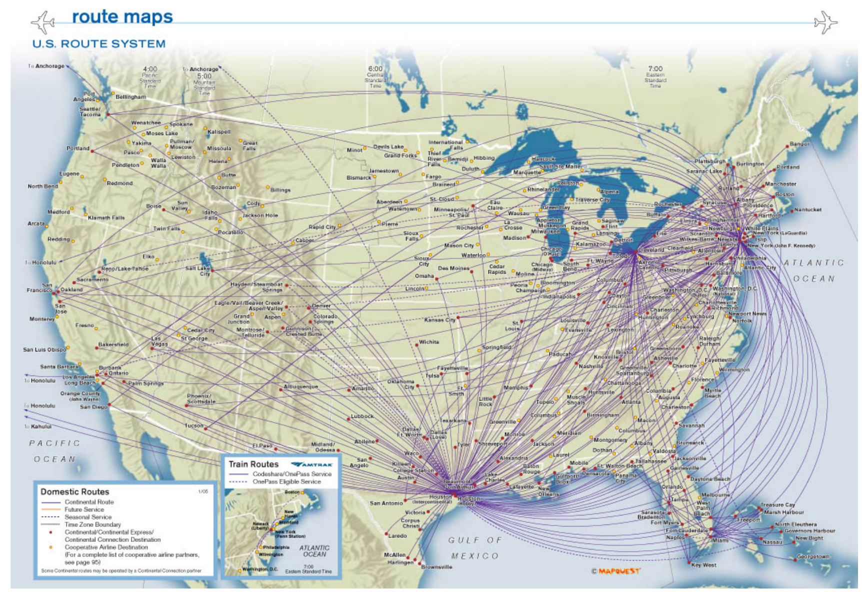
EXAMPLE 2: RAIL (SUBWAY) TRANSPORATION

London



Stockholm

EXAMPLE 3: AIR TRANSPORTATION



TRANSPORATION NETWORKS AND THEIR REPRESENATIONS

- Nodes (vertices) for connecting points
  - Flow conservation, capacity and delay
- Links (arcs, edges) for routes
  - Capacity, cost (travel time), flow propagation
- Degree of a node, path and connectedness
- A node-node adjacency or node-link incidence matrix for network structure

Characteristics of transportation networks

- Highway networks
  - Nodes rarely have degrees higher than 4
  - Many node pairs are connected by multiple paths
  - Usually the number of nodes < number of links < number of paths in a highway network
- Air route networks
  - Some nodes have much higher degrees than others (most nodes have degree one)
  - Many node pairs are connected by a unique path
- Urban rail networks
  - Falls between highway and air networks

Flows in a Highway Network

$N:$ set of nodes

$A:$ set of links

$I:$ set of origins

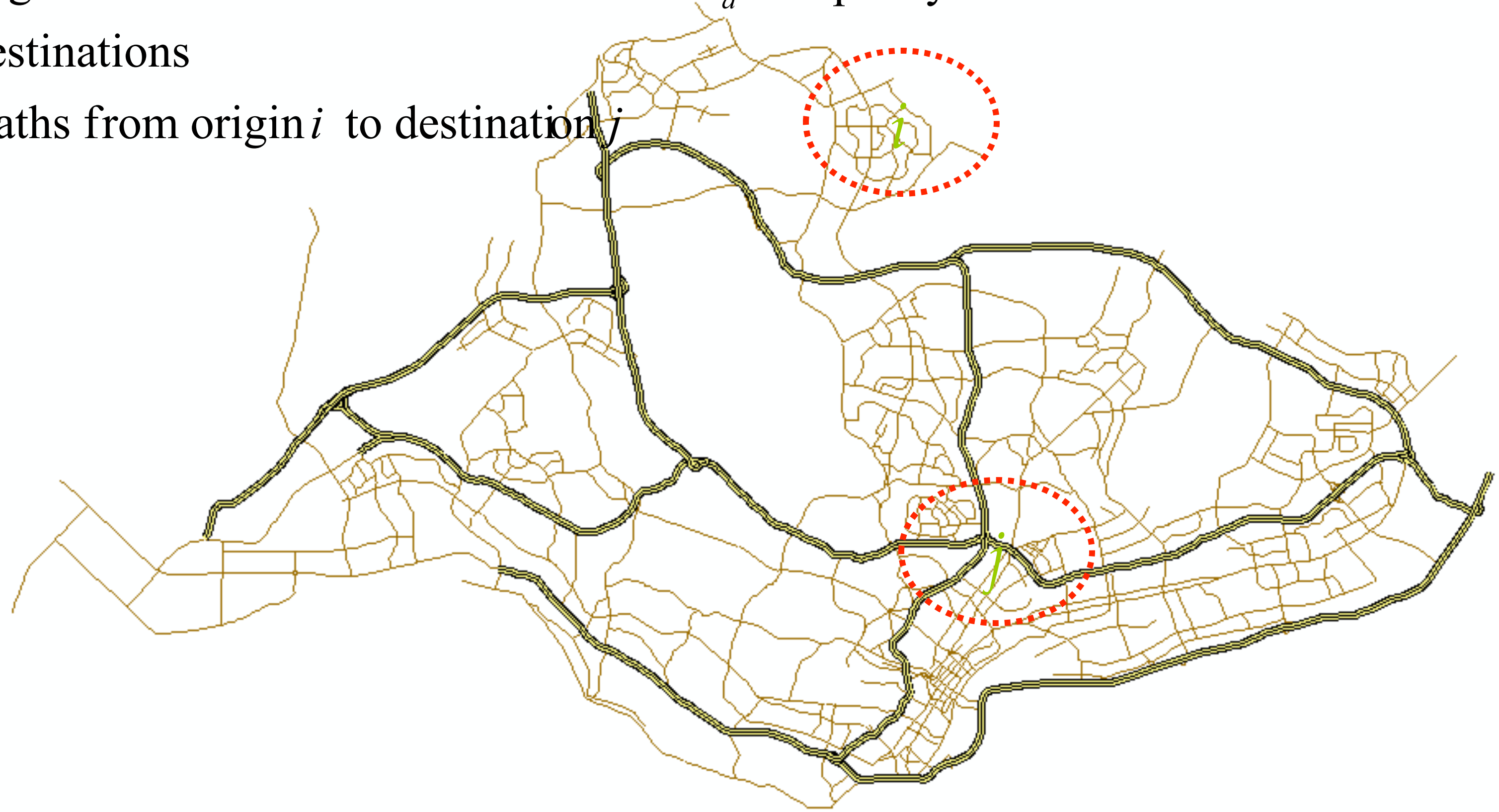
$J:$ set of destinations

$R_{ij}:$ set of paths from origin i to destination j

$t_a(v_a, C_a):$ link travel cost function

$q_{ij}:$ Traffic demand from origin i to destination j

$C_a:$ Capacity on link a

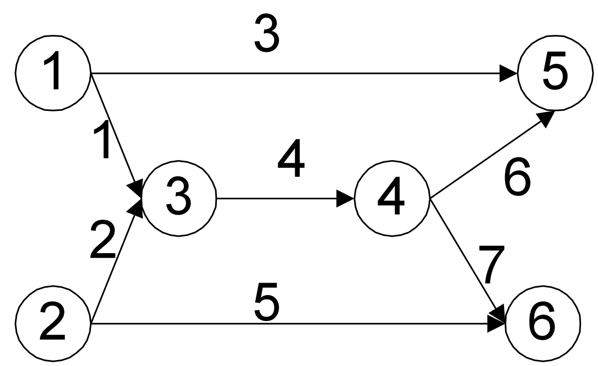


Flows in a Highway Network (Cont'd)

- Path flows: $\left\{ f_r^{ij}, r \in R_{ij}, i \in I, j \in J \right\}$
  - Flow conservation equations

$$\sum_{r \in R_{ij}} f_r^{ij} = q_{ij}, i \in I, j \in J$$

$$f_r^{ij} \geq 0$$



  - Set of feasible path flows

$$S = \left\{ f = \left(\cdots, f_r^{ij}, \cdots \right)^T \middle| \sum_{r \in R_{ij}} f_r^{ij} = q_{ij}; f_r^{ij} \geq 0; r \in R_{ij}, i \in I, j \in J \right\}$$

Flows in a Highway Network (Cont'd)

- Origin based link flows: $\left\{v_a^i, a \in A, i \in I\right\}$
  - Flow conservation equations

$$\sum_{a \in A_i^-} v_a^i = \sum_{j \in J} q_{ij}, i \in I$$

$$\sum_{a \in A_j^+} v_a^i = q_{ij}, j \in J$$

$$\sum_{a \in A_n^+} v_a^i - \sum_{a \in A_n^-} v_a^i = 0, n \in N \setminus \left\{I \cup J\right\}$$

$$v_a^i \geq 0, i \in I, a \in A$$

$$A_n^- = \left\{\text{all links entering node } n\right\}, n \in N$$

$$A_n^+ = \left\{\text{all links leaving node } n\right\}, n \in N$$

  - Set of feasible origin based link flows

$$S^I = \left\{ v^I = \left(\cdots, v_a^i, \cdots\right)^T \middle| \left\{v_a^i, i \in I, a \in A\right\} \text{ satisfies the above equations} \right\}$$

Flows in a Highway Network (Cont'd)

- Link flows: $v = (\cdots, v_a, \cdots)^T$
  - Set of feasible link flows

$$\Omega = \left\{ v \middle| \; v_a = \sum_{i \in I} \sum_{j \in J} \sum_{r \in R_{ij}} f_r^{ij} \delta_{ar}^{ij}, a \in A; \sum_{r \in R_{ij}} f_r^{ij} = q_{ij}; f_r^{ij} \geq 0; r \in R_{ij}, i \in I, j \in J \right\}$$

where

$$\delta_{ar}^{ij} = \begin{cases} 1, & \text{if path } r \in R_{ij} \text{ using link } a \\ 0, & \text{otherwise} \end{cases}$$

It is a convex, closed and bounded set

Costs in a Highway Network (Cont'd)

- Travel cost on a path $r \in R_{ij}, i \in I, j \in J$

$$c_r^{ij} = \sum_{a \in A} t_a(v_a) \delta_{ar}^{ij}, r \in R_{ij}, i \in I, j \in J$$

- The shortest path from origin *i* to destination *j*

$$\mu_{ij} = \min_{r \in R_{ij}} \left\{ c_r^{ij} \right\}, i \in i, j \in J$$

- Total system travel cost

$$\sum_{a \in A} t_a(v_a) v_a$$

Behavioral Assumptions

Act on self interests (User Equilibrium):

- Travelers have full knowledge of the network and its traffic conditions
- Each traveler minimizes his/her own travel cost (time)

Act on public interests (System Optimal):

- Travelers choose routes to make the total travel time of all travelers minimal (which can be achieved through choosing the routes with minimal marginal travel cost)

$$\min \sum_{a \in A} t_a(v_a) v_a$$

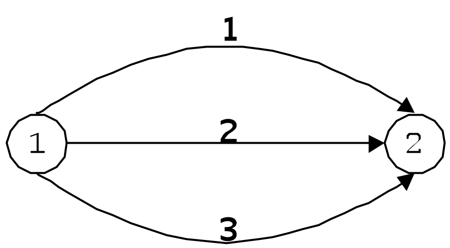
THE USER EQUILIBRIUM CONDITION

- At UE, no traveler can unilaterally change his/her route to shorten his/her travel time (Wardrop, 1952). It's a Nash Equilibrium. Or
- At UE, all paths connecting an origin-destination pair that carry flow must have minimal and equal travel time for that O-D pair $f_r^{ij}\left(c_r^{ij} - \mu_{ij}\right) = 0, \quad \left(c_r^{ij} - \mu_{ij}\right) \geq 0, \quad f_r^{ij} \geq 0$
- However, the total travel time for all travelers may not be the minimum possible under UE.

A special case: no congestion, infinite capacity

- Travel time is independent of flow intensity
- UE & SO both predict that all travelers will travel on the shortest path(s)
- The UE and SO flow patterns are the same

$$\left\{ f_r^{ij}, r \in R_{ij}, i \in I, j \in J \right\}$$



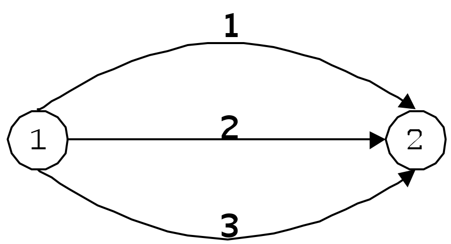
i=1, j=2, q_{12}=12
t1=1
t2=10
t3=40

→

V1=12, f1=12, μ=1
V2=0, f2=0
V3=0, f3=0

We can check the UE and SO conditions

A case with congestion



Link cost functions:

$t_1(v) = 1 + v_1$

$t_2(v) = 1 + v_2 + \frac{1}{2} v_1$

$t_3(v) = 40$

O-D demand:

$q_{12} = 12$

UE path flow pattern:

$f_1^{12*} = 8, f_2^{12*} = 4, f_3^{12*} = 0$

UE origin based link flow pattern

$v_1^{1*} = 8.0, v_2^{1*} = 4.0, v_3^{1*} = 0.0$

UE link flow pattern:

$v_1^* = 8, v_2^* = 4, v_3^* = 0$

Path travel cost pattern:

$c_1^{12*} = 9, c_2^{12*} = 9, c_3^{12*} = 40$

UE O-D travel cost:

$\mu_{12} = 9$

$$\sum_i v_i t_i = 108$$

SO:

$$\min v_1 t_1(v) + v_2 t_2(v) + v_3 t_3(v)$$
$$v_1 + v_2 + v_3 = 12$$
$$v_1, v_2, v_3 \geq 0$$

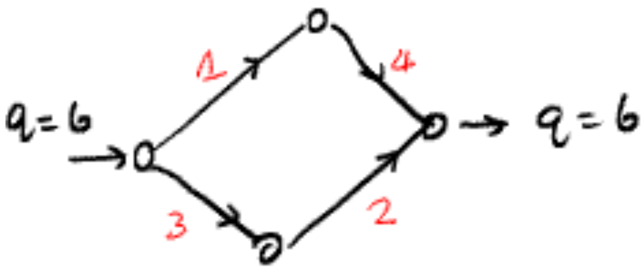
$\longrightarrow$

$v_1 = 6, t_1 = 7$

$v_2 = 6, t_2 = 10$

$v_3 = 0, t_3 = 40 \quad \sum_i v_i t_i = 102$

The Braess' Paradox



$t_1 = 50 + x_1$

$t_2 = 50 + x_2$

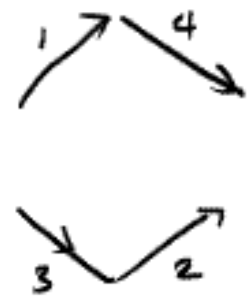
$t_3 = 10\, x_3$

$t_4 = 10\, x_4$

UE sol'n

2 paths

flow:



$f_1 = 3$

$f_2 = 3$

+ = 6

travel time

$c_1 = t_1 + t_4 = 53 + 30 = 83$

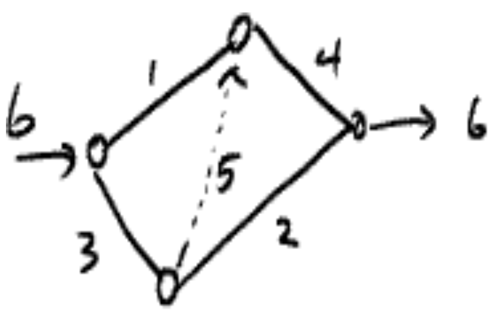
$c_2 = t_3 + t_2 = 30 + 53 = 83$

total trav. time = $\sum x_i \cdot t_i = 166$

Now, if we add an additional link to the network with the following cost function $t_5 = 10 + x_5$

The Braess' Paradox-Cont.

Now, if we add an additional link to the network with the following cost function $t_5 = 10 + x_5$



Now we have a new path

with γ flow

so $c_3 = t_3 + t_5 + t_4 = 0 + 10 + 0 = 10$

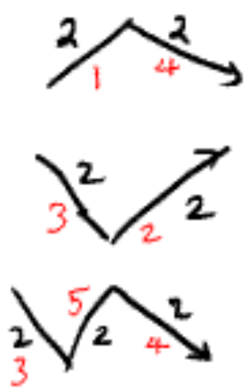
$< c_2$ & c_1

traffic is no longer in equilibrium → new equilibrium will be produced.

The Braess' Paradox-Cont.

By inspection, we shift 1 unit of flow from path 1 & 2, resp. to path 3

path



1 $\quad x_1 = 2$

2 $\quad x_2 = 2$

3 $\quad x_3 = 4$

$x_4 = 4$

$c_1 = t_1 + t_4 = 52 + 40 = 92$

$c_2 = t_2 + t_3 = 52 + 40 = 92$

$c_3 = t_3 + t_5 + t_4 = 40 + 12 + 40 = 92$

a new UE state

but the path travel times are higher (92 vs. 83)

total travel time $= \sum t_i \cdot x_i = 276 > 166$

General Cases for UE:

$$\min_{v} h(v) = \sum_{a \in A} \int_0^{v_a} t_a(\omega) d\omega$$

subject to

$$\sum_{r \in R_{ij}} f_r^{ij} = q_{ij}, i \in I, j \in J$$

$$v_a = \sum_{i \in I} \sum_{j \in J} \sum_{r \in R_{ij}} f_r^{ij} \delta_{ar}^{ij}, a \in A$$

$$f_r^{ij} \geq 0, r \in R_{ij}, i \in I, j \in J$$

With an increasing travel time function, this is a strictly (nonlinear) convex minimization problem.

It can be shown that the KKT condition of the above problem gives precisely the UE condition

The relation between UE and SO

UE

$$\min_{v\in\Omega} h(v) = \sum_{a\in A} \int_0^{v_a} t_a(\omega)\,d\omega$$

$$\hat{t}_a(v_a) = t_a(v_a) + \boxed{v_a \frac{dt_a}{dv_a}}$$

$\downarrow$

$$\min_{v\in\Omega} \hat{h}(v) = \sum_{a\in A} \int_0^{v_a} \hat{t}_a(\omega)\,d\omega$$

SO

$$\min_{v\in\Omega} H(v) = \sum_{a\in A} v_a t_a(v_a)$$

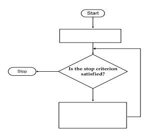
$$\breve{t}_a(v_a) = \frac{1}{v_a} \int_0^{v_a} t_a(\omega)\,d\omega$$

$\downarrow$

$$\min_{v\in\Omega} \breve{H}(v) = \sum_{a\in A} v_a \breve{t}_a(v_a)$$

Algorithms for for Solving the UE Problem

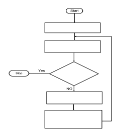
- Generic numerical iterative algorithmic framework for a minimization problem



Yes

No

Algorithms for Solving the UE Problem (Cont'd)



Applications

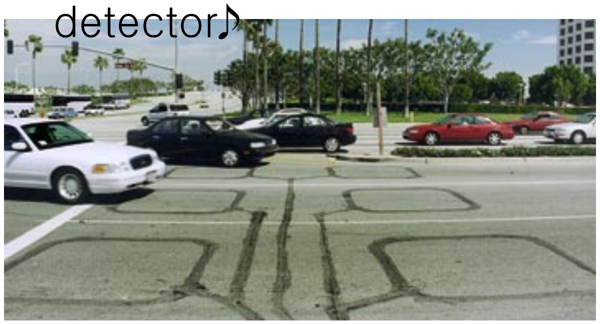
- Problems with thousands of nodes and links can be routinely solved
- A wide variety of applications for the UE problem
  - Traffic impact study
  - Development of future transportation plans
  - Emission and air quality studies

Intelligent Transportation System (ITS) Technologies

- On the road
- Inside the vehicle
- In the control room

“EYES” OF THE ROAD

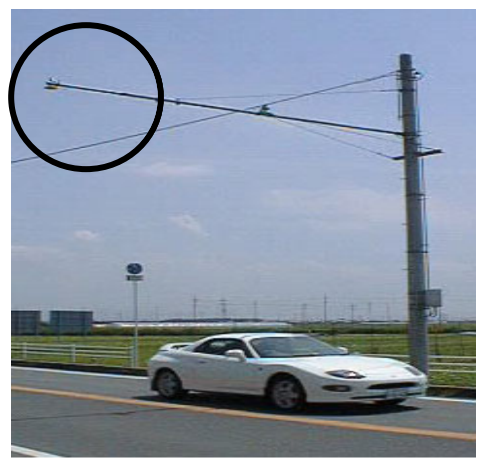
– Loop detector♪



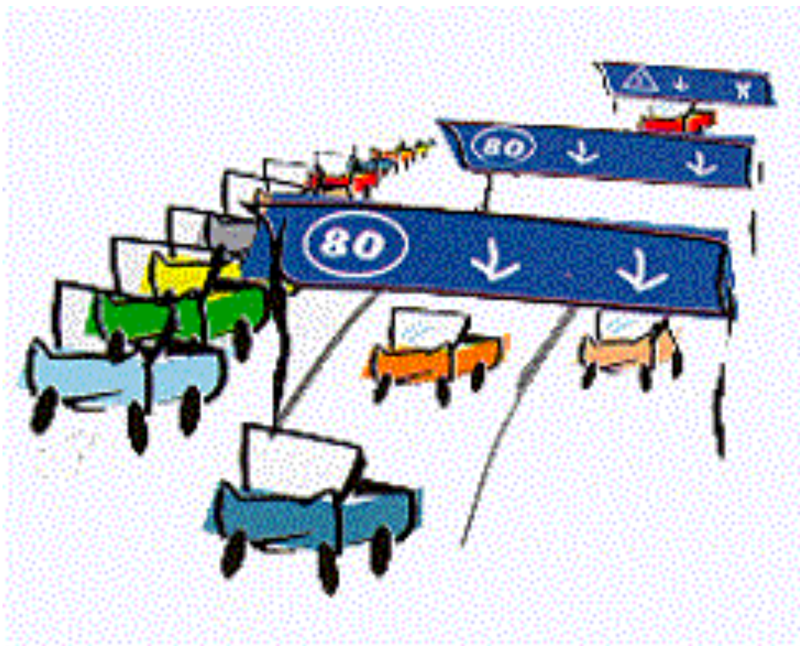
– Video detector

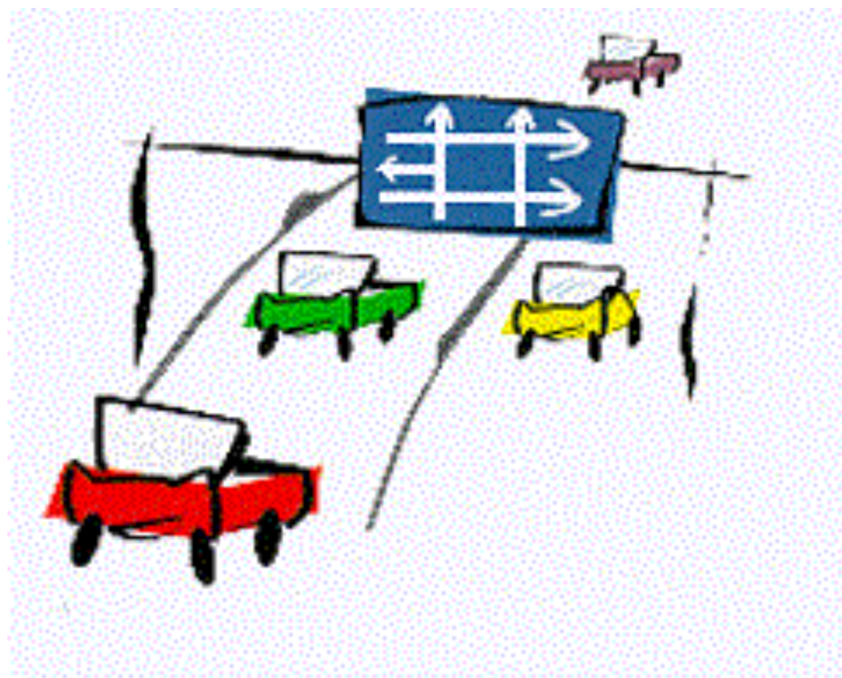
– Infrared detector♪

– Ultrasonic detector♪



SMART ROADS





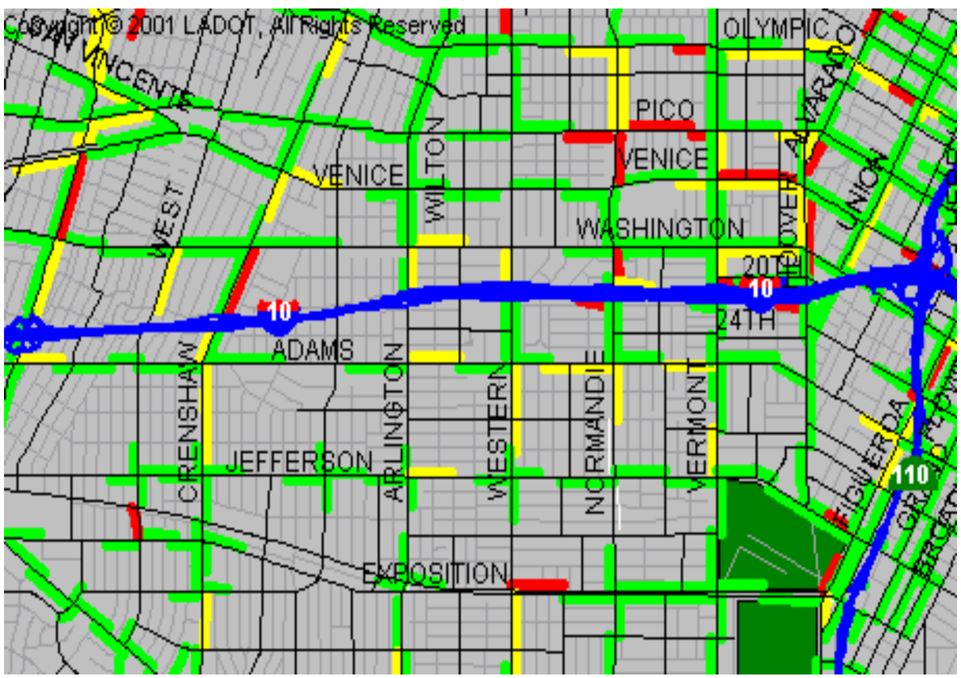
SMART VEHICLES

SAFETY, TRAVEL SMART GAGETS, MOBILE OFFICE(?)

SMART PUBLIC TRANSIT

- GPS + COMMUNICATIONS FOR
  - BETTER SCHEDULING & ON-TIME SERVICE
  - INCREASED RELIABILITY
- COLLISION AVOIDANCE FOR
  - INCREASED SAFETY

SMART CONTROL ROOM



If you wish to learn more about urban traffic problems

- ECI 256: Urban Congestion and Control (every Fall)
- ECI 257: Flows in Transportation Networks (Winter)