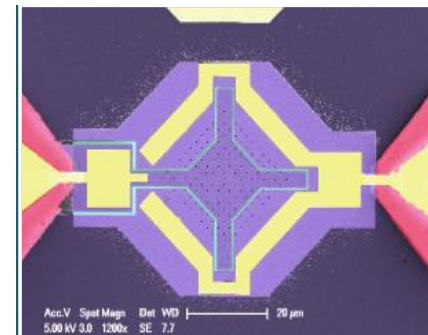
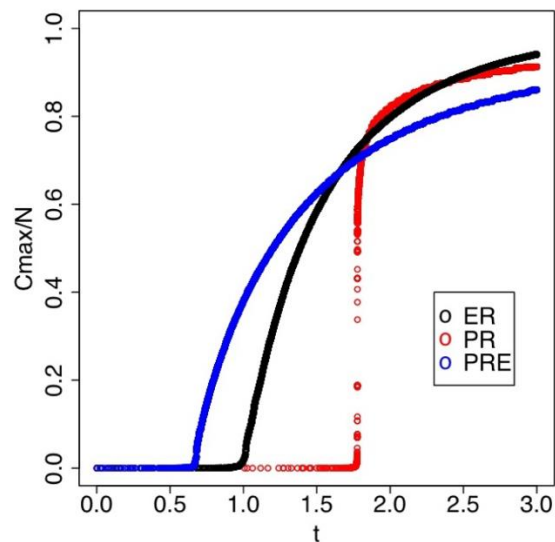
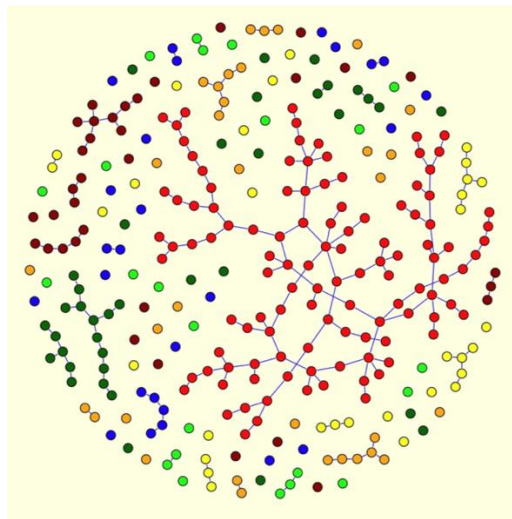


Complex networks with complex nodes

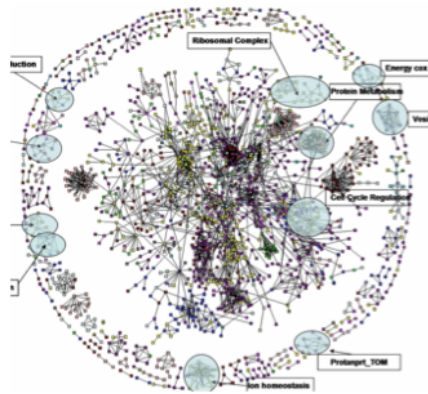


Raissa M. D'Souza

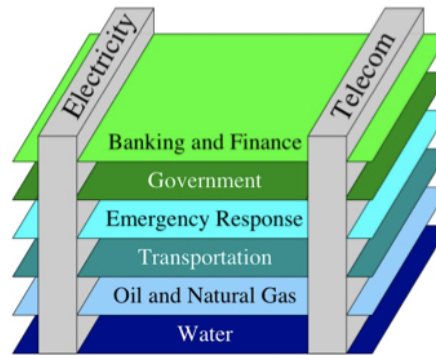
University of California, Davis



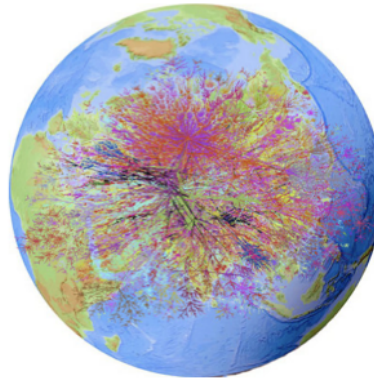
Structure and function of interdependent networks



**Biological & Ecological
networks**



Critical Infrastructure



**Information and Communication
technology**

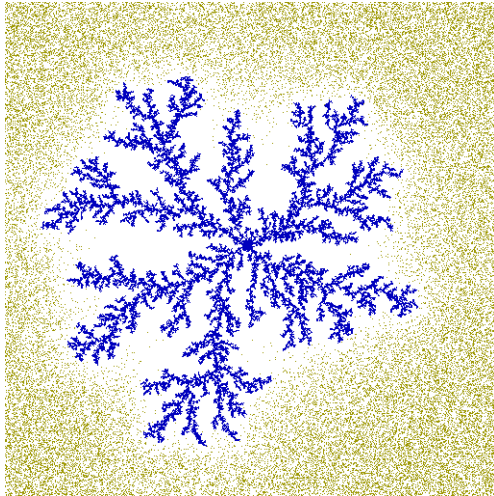


**Social networks:
Economics & Epidemics**

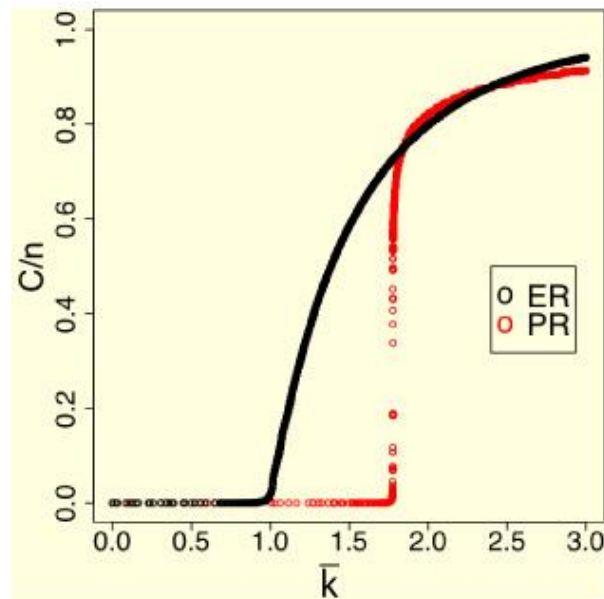
Each network is a complex system with emergent behaviors

What are emergent collective behaviors?

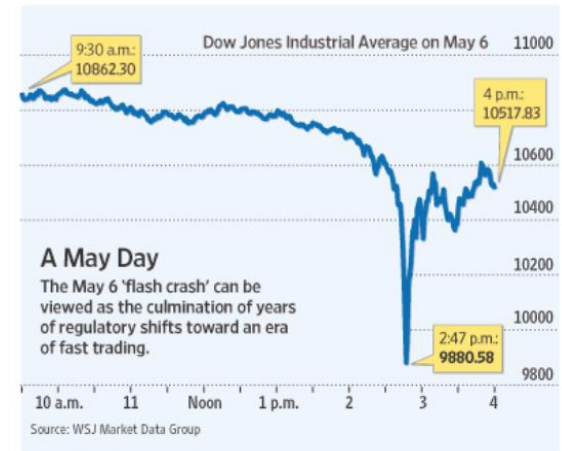
Behaviors not predicted a priori from the constituent equations of motion.



Synchronization and pattern formation.



Phase transitions
“Tipping points”

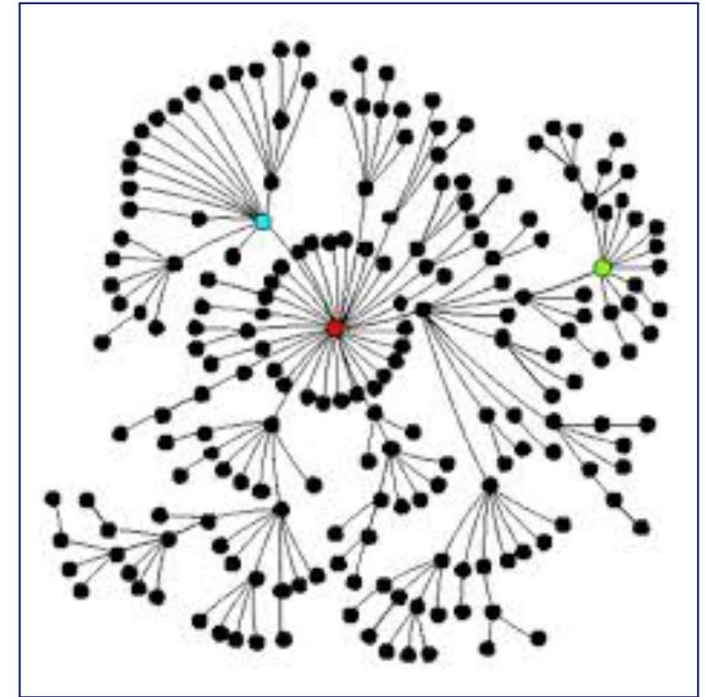
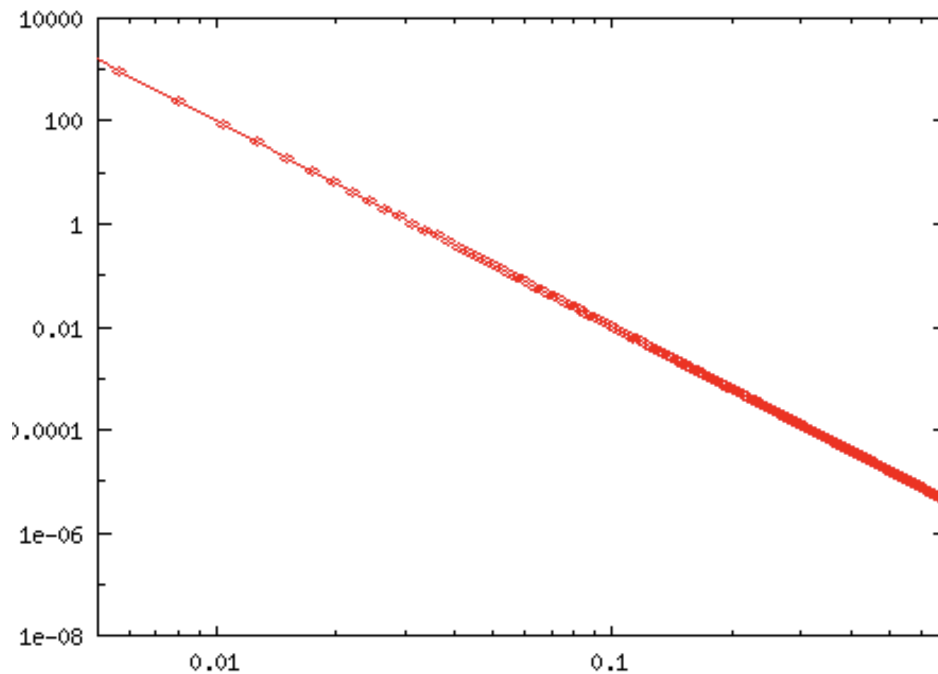


Cascading failures.

Statistical physics approach to networks

Start with a “random graph”

The Degree distribution, p_k
(Fraction of nodes of degree k)

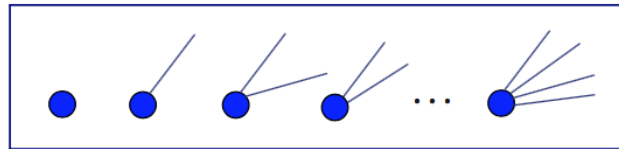
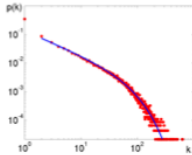


A power-law distribution is a “scale-free” network

Probabilistic properties of the ensemble of graphs.

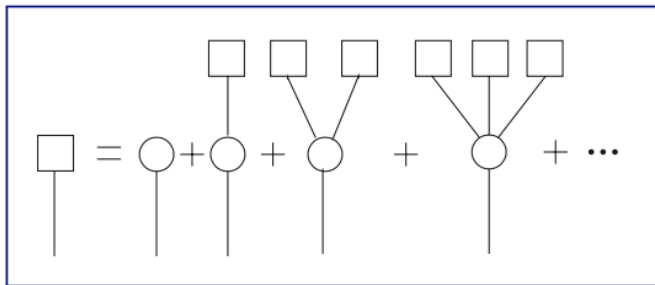
Calculating properties of the random graph

- **Configuration models** (Bollobás 1980, Molloy and Reed *RSA* 1995).
Enumerating over all networks with specified degree distribution $\{p_k\}$.



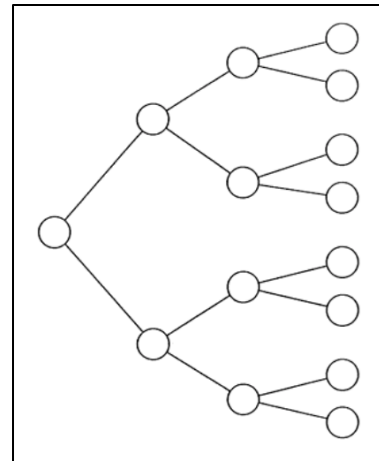
Start with half-edges.
Assign a random matching
to create an instance.

- **Generating functions :** $G_0(x) = \sum_k p_k x^k$



$$p_c = \frac{1}{g'_1(1)} = \frac{\langle k \rangle}{\langle k^2 \rangle - \langle k \rangle}.$$

Criteria for giant component



ing properties →

Caveat:
Requires
locally tree-
like structure

Properties of the random graph

- **Rate equations / Kinetic theory** :
Mean-field evolution of clusters/
graph structures.

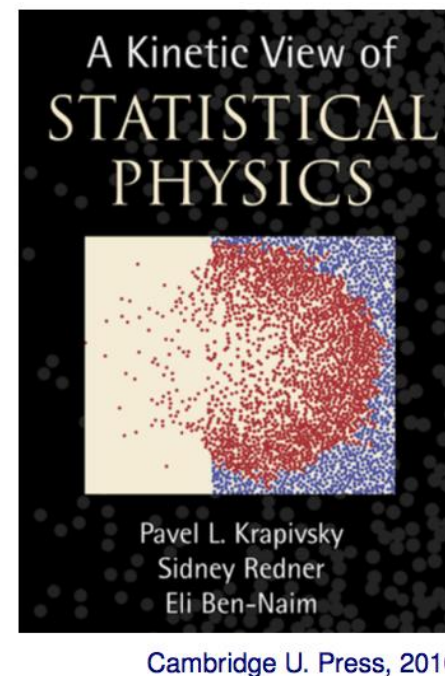
$$x_k(t+1) = F(\vec{x}(t))$$

e.g., “preferential attachment”:

$x_k \equiv$ fraction of nodes of degree k :

$$x_k(t+1) = \frac{k-1}{n} x_{k-1}(t) - \frac{k}{n} x_k(t)$$

Analyzed for asymptotic properties: $N \rightarrow \infty$ and $t \rightarrow \infty$.

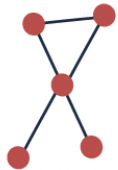


Achievements of random graphs

- **Vulnerability** to “hub” removal / **resilience** to random removal for broad-scale degree distributions.
- Epidemic spreading
- Epidemic threshold can approach zero!
- Percolation and extent of connectivity
- Critical thresholds for cascades
- Diffusion and spreading
- Opinion dynamics, voter models, etc

Statistical physics of networks

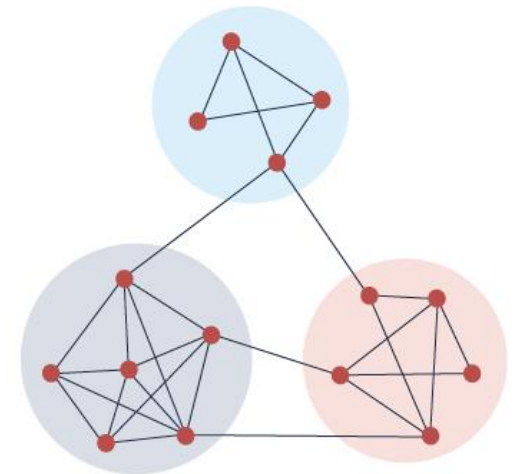
- Thermodynamic limit $N \rightarrow \infty$
 - Equilibrium / Steady-state behavior
 - Ensemble properties
-
- Simple nodes/edges (e.g., often binary state)
 - Non-trivial network structure, “complex networks”:
 - Broad-scale degree distributions,



- Clustering/
triangular closure



- Small-worlds



- Community structure

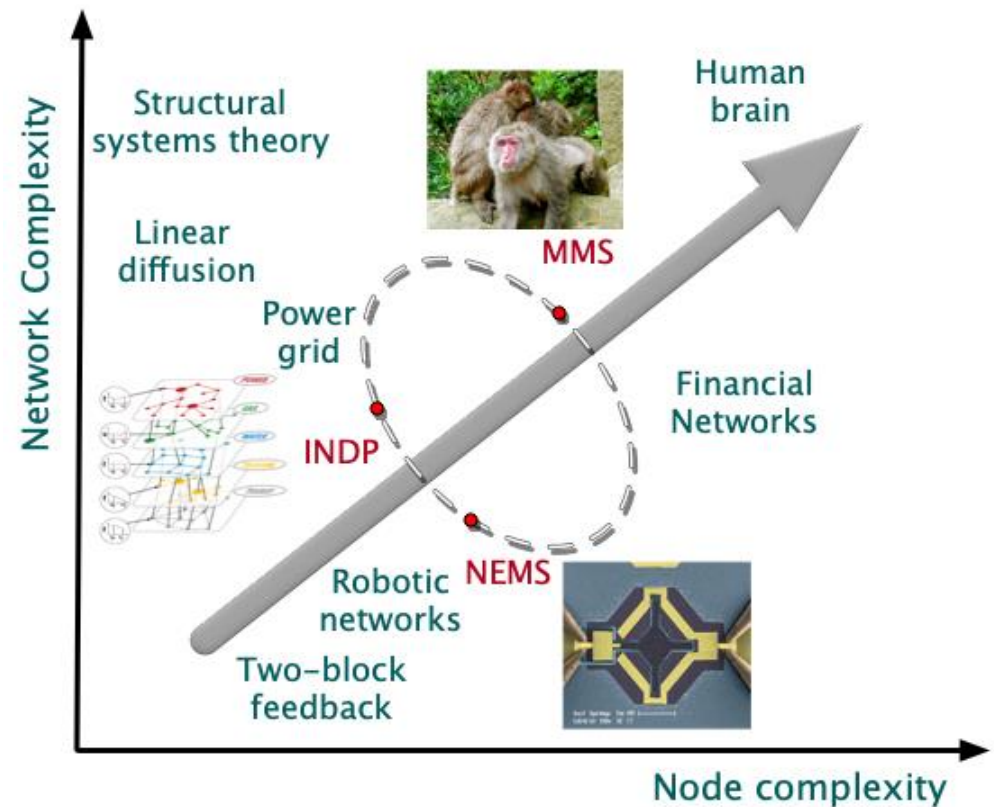
Origin of emergent collective behaviors?

Where is the complexity?

In the network structure?

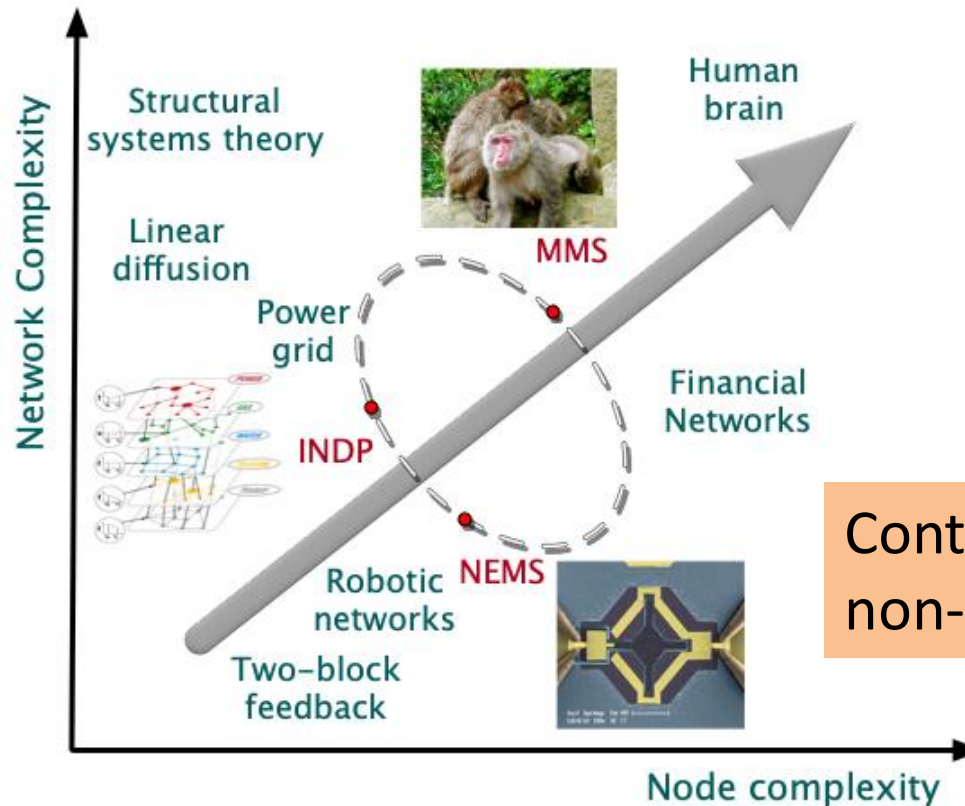
In the nodal dynamics?

In both?



Complex networks with complex nodes

Statistical
Physics



Control theory and
non-linear dynamics



ARO FY2013 MURI TOPIC #7

Submit white papers and proposal to Army Research Office

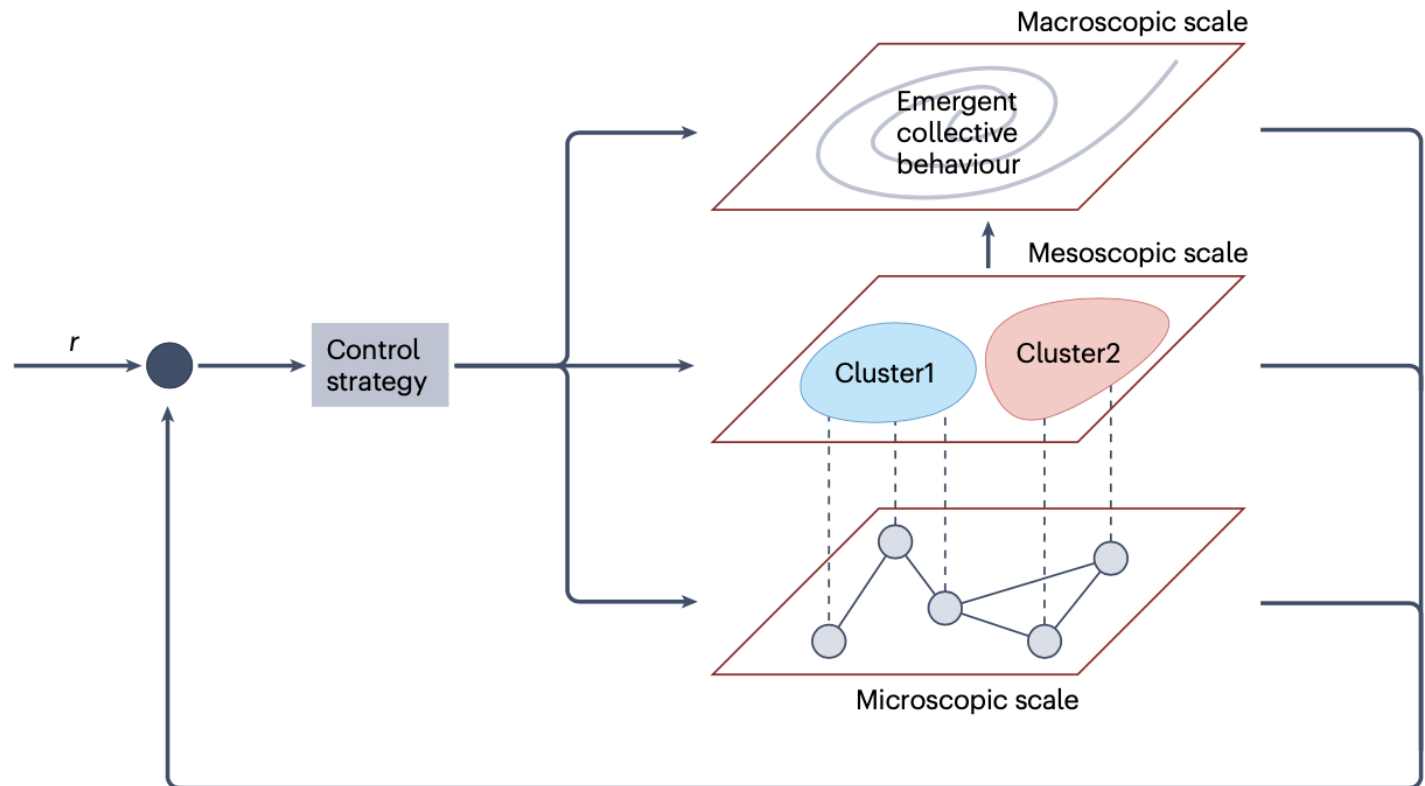
Controlling Collective Phenomena in Complex Networks

Thank you to the MURI team W911NF-13-1-0340.

Controlling complex networks with complex nodes

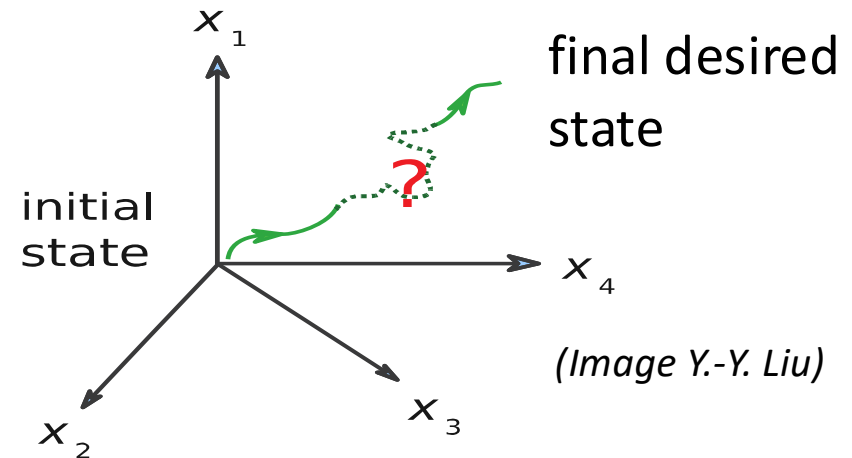
Published online: 24 March 2023

Raissa M. D'Souza ^{1,2,3} , Mario di Bernardo ^{4,5}  & Yang-Yu Liu ^{6,7} 

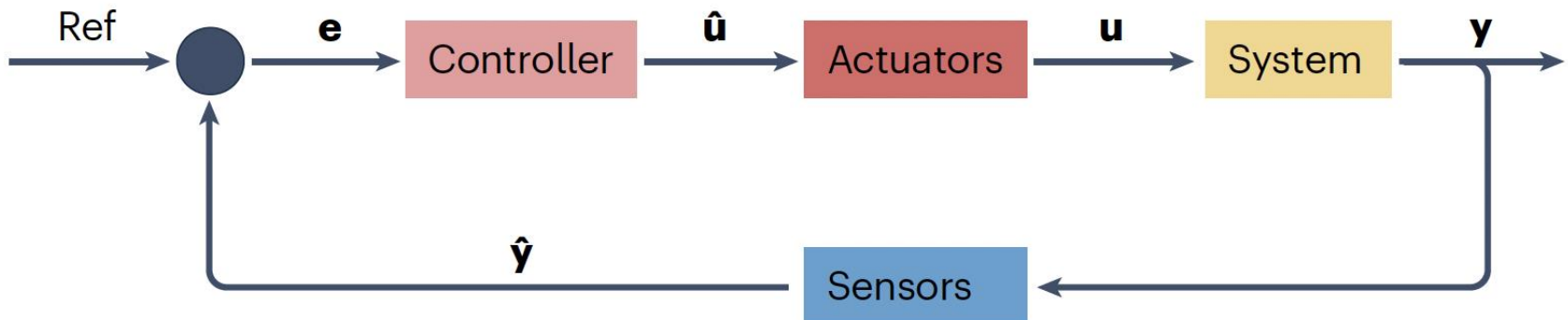


Classic feedback control theory

Control: Drive the system from any initial starting configuration to any specified end configuration in finite time.



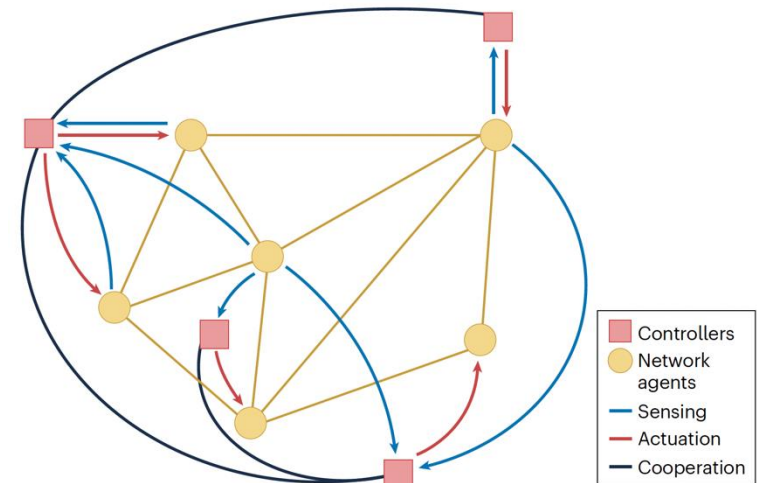
Feedback control paradigm:



Classic control tools are not enough

- Millions of degrees of freedom
 - Full control is not feasible
 - Full knowledge may not be

Network control



- Self-organizing systems
 - Distributed control
 - Cascading failures – SOC and DKs are prevalent
 - Unanticipated response to interventions
 - e.g., Noel, Brummitt, RD., *Phys. Rev. Lett.* **111** (2013)

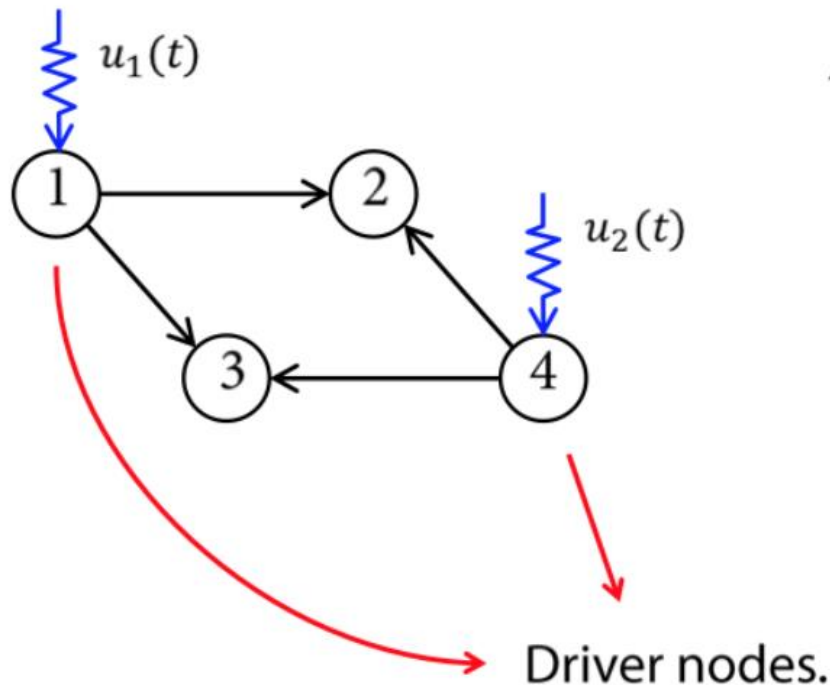
What do we want to control?

- Every degree of freedom?
- Macroscopic or microscopic details? E.g.,
 - How many people are infected (macroscopic)
 - Which particular people are infected (microscopic).
- Steer towards some class of behaviors? Avoid certain attractors.

Starting point: Linear time invariant

Linear time-invariant system:

$$x(t + 1) = \mathbf{A} x(t) + \mathbf{B} u(t)$$



$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & a_{24} \\ a_{31} & 0 & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} b_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & b_2 \end{pmatrix}$$

$\mathbf{B}$ is a $M \times N$ matrix

- N nodes
- M control signals/driver nodes
- Columns are the unit vector of each driver node

We want to find N_D the minimum number of driver nodes

Control of Linear time invariant (LTI) systems

Controllability matrix: $C = [B \ AB \ A^2B \ \dots \ A^{n-1}B]$

Kalman's controllability criteria: $\text{rank } C = N$

Controllable!

The matrix C is full-row rank (each row linearly independent)

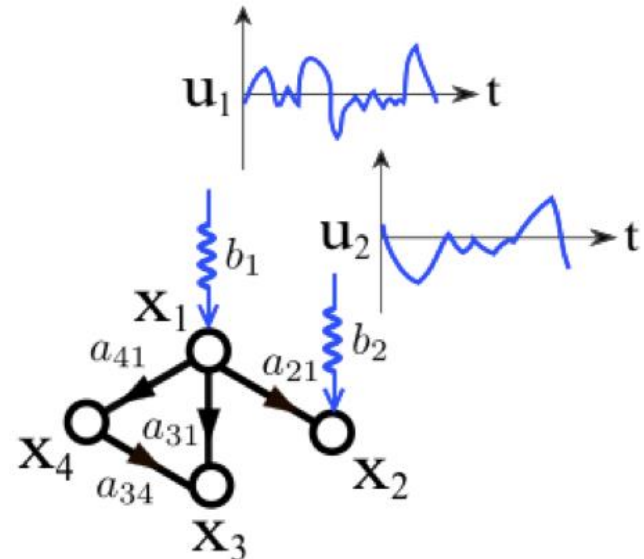
Kalman, *J.S.I.A.M. Control* (1963)

The controllability Gramian

$$W_c(t) = \int_{t_0}^t e^{A(t-\tau)} B B^* e^{A^*(t-\tau)} d\tau.$$

If and only if the pair (A,B) is controllable, W_c is n the minimum energy control signal:

$$u(t) = -B^* e^{A^*(t_1-t)} W_c^{-1}(t_1) [e^{A(t_1-t_0)} x_0 - x_1].$$



Structural control of LTI systems

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. AC-19, NO. 3, JUNE 1974

201

Structural Controllability

CHING-TAI LIN, MEMBER, IEEE

Abstract—The new concepts of “structure” and “structural controllability” for a linear time-invariant control system (described by a pair (A, b)) are defined and studied. The physical justification of these concepts and examples are also given.

The graph of a pair (A, b) is also defined. This gives another way of describing the structure of this pair. The property of structural controllability is reduced to a property of the graph of the pair (A, b) . To do this, the basic concept of a “cactus” and the related concept of a “precactus” are introduced. The main result of this paper states that the pair (A, b) is structurally controllable if and only if the graph of (A, b) is “spanned by a cactus.” The result is also expressed in a more conventional way, in terms of some properties of the pair (A, b) .

ing entry of (Ab) is also fixed (zero). Then one defines the pair (A_0, b_0) to be *structurally controllable* if and only if there exists a completely controllable pair (A, b) which has the same structure as (A_0, b_0) .

The concept of “structural controllability” of a pair (A_0, b_0) makes the meaning of controllability (in the usual sense) more complete from the physical point of view. In fact, it is preferred whenever (A_0, b_0) represents an actual physical system (that involves parameters only approximately determined). Actually, the completely controllable pair (A, b) can be considered as “physically



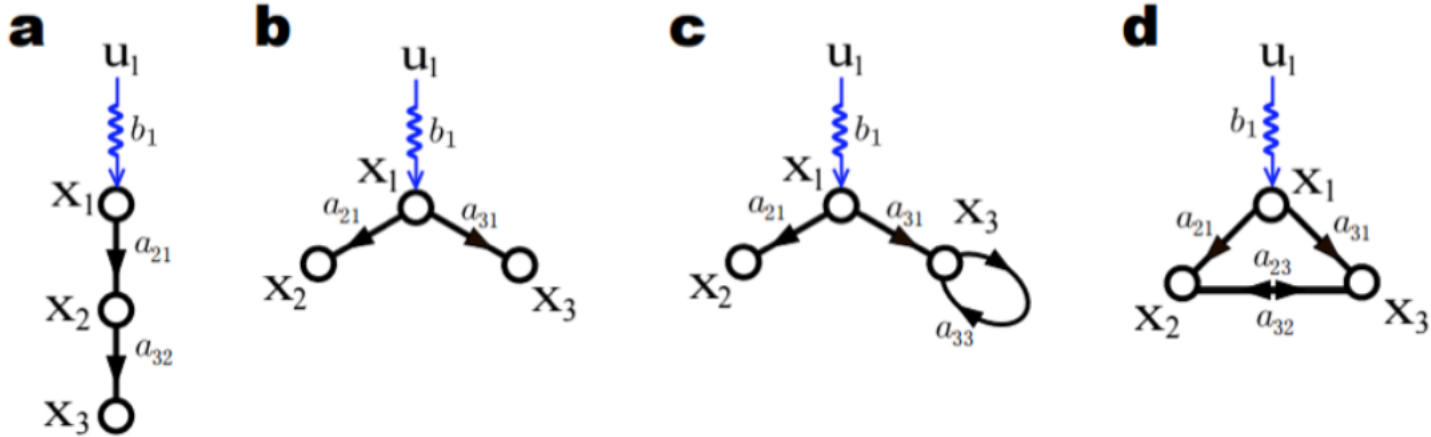
C-T Lin, *IEEE Trans on Automatic Control*, 1974

We treat the nonzero elements in A and B as free parameters, and we keep the zero entries fixed.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & 0 & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{21}^* & 0 & a_{23}^* & a_{24}^* \\ a_{31}^* & 0 & 0 & a_{34}^* \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Examples

Loops self-regulate:



$$\mathbf{C} = [\mathbf{B}, \mathbf{A} \cdot \mathbf{B}, \mathbf{A}^2 \cdot \mathbf{B}]$$

$$b_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{21} & 0 \\ 0 & 0 & a_{32}a_{21} \end{pmatrix}, \quad b_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{21} & 0 \\ 0 & a_{31} & 0 \end{pmatrix}, \quad b_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{21} & 0 \\ 0 & a_{31} & a_{33}a_{31} \end{pmatrix}, \quad b_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{23}a_{31} \\ 0 & a_{31} & a_{32}a_{21} \end{pmatrix}$$

rank $\mathbf{C} = N=3$
controllable

rank $\mathbf{C}=2 < N=3$
uncontrollable

rank $\mathbf{C} = N=3$
controllable

rank $\mathbf{C} = ?$
controllable ?

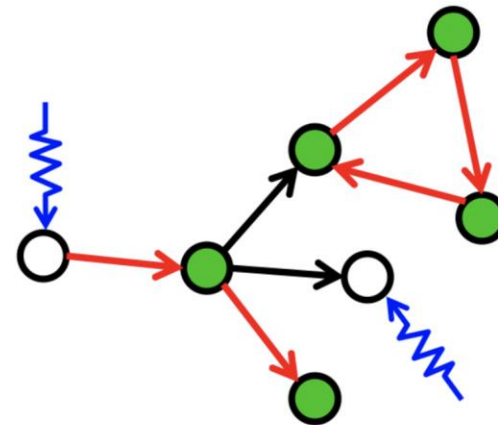
Structural control meets maximum matching



Y.-Y. Liu, A.L. Barabasi, J.-J. Slotine
“Controllability of complex networks”
Nature 2011.

Matching in Directed Networks

The unmatched nodes are the driver nodes.



Cavity method from spin glasses

GOOD: algorithms and analytical tools!

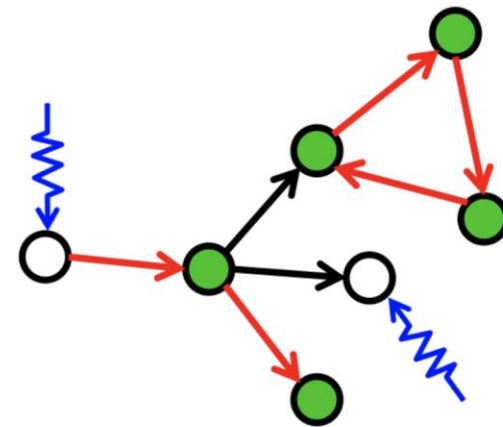
Structural control meets maximum matching

Matching in Directed Networks

Cavity method from spin glasses

See details in Lec 10 slides,
starting on page 29

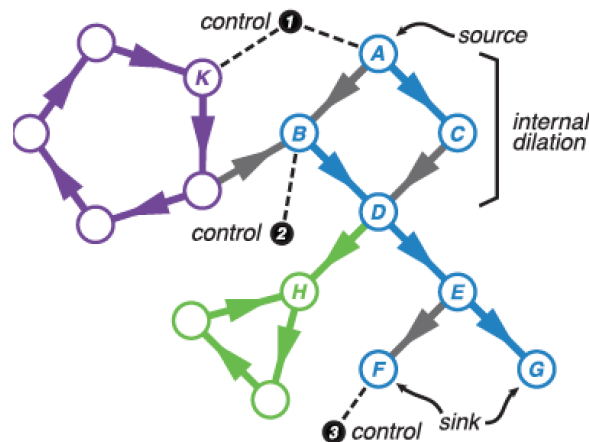
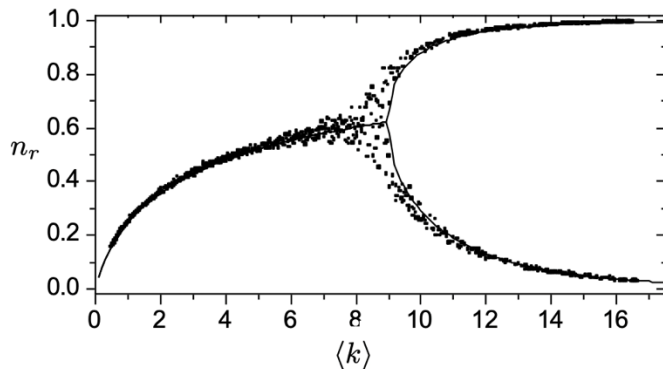
The unmatched nodes are the driver nodes.



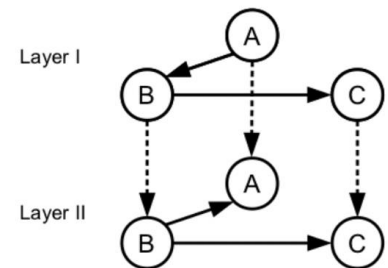
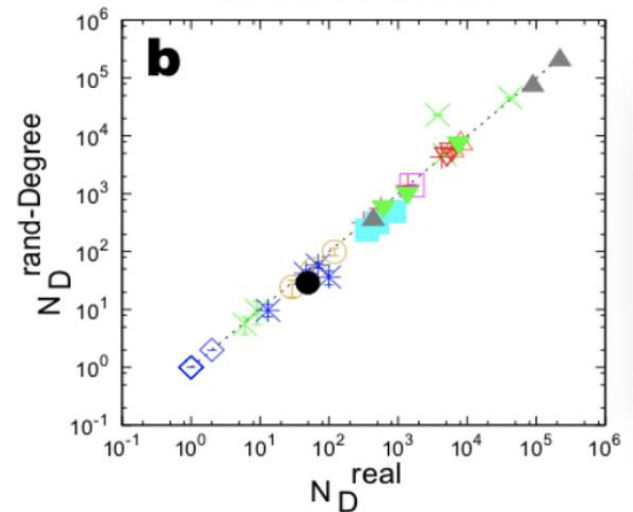
GOOD: algorithms and analytical tools!

Insights from structural control:

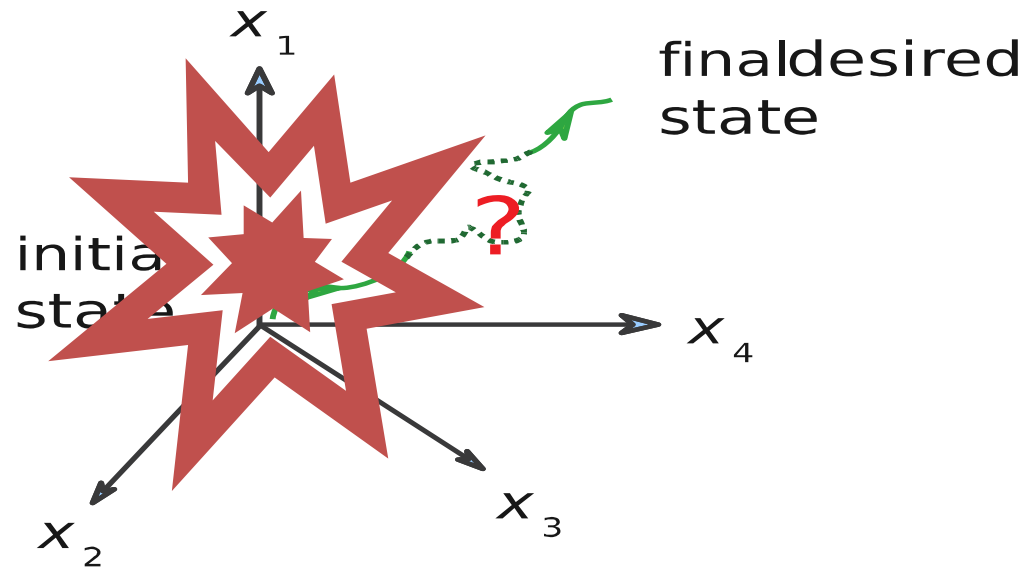
- Role of degree distribution
- Control energy
- Exact controllability
- Connection to core percolation
- Control profiles: sources, sinks, dilations
- Control of multiplex LTI networks
- Target control



Degree-preserving randomization



Real-world: Massive, non-linear systems



(Image Y.-Y. Liu)

- Possibly only partial knowledge
- Non-linear nodal dynamics
- Self-organization / far from equilibrium

Part II.

Non-linear dynamics
and “control of chaos”

Non-linear dynamics

Start from the dynamical equations of motion – complex dynamics, low dimensional

VOLUME 64, NUMBER 11

PHYSICAL REVIEW LETTERS

12 MARCH 1990

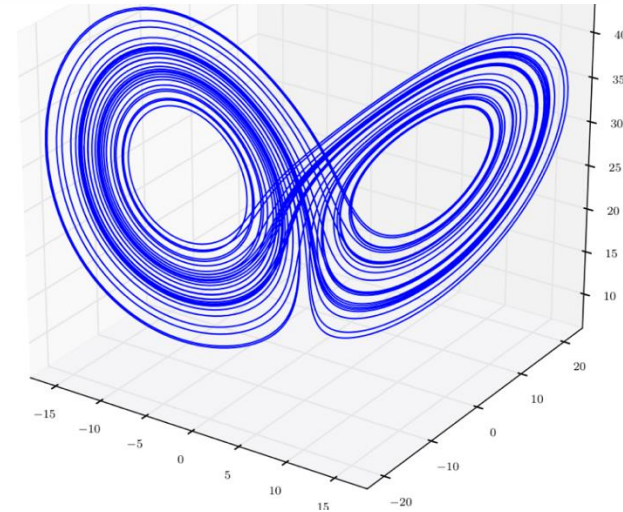
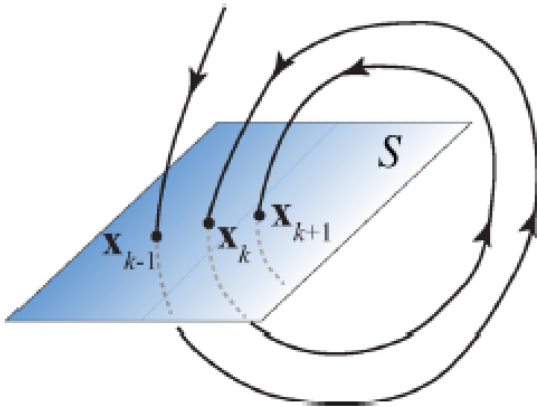
Controlling Chaos

Edward Ott,^{(a),(b)} Celso Grebogi,^(a) and James A. Yorke^(c)

University of Maryland, College Park, Maryland 20742

(Received 22 December 1989)

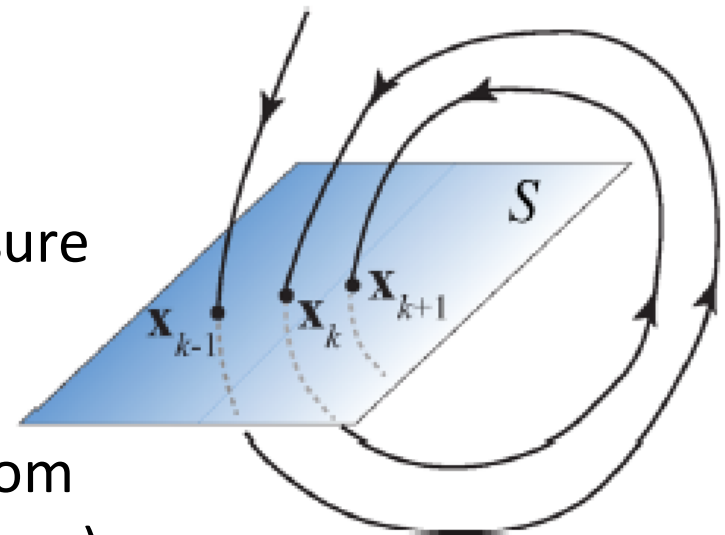
- An infinite number of unstable periodic orbits typically embedded in a chaotic attractor.
- $\frac{dx}{dt} = F(\mathbf{x}, p)$, where $\mathbf{x}$ is a **3D vector**



- Stabilize desired orbit by making only small time-dependent perturbations of a system parameter, p

Controlling chaos (OGY theory, cont)

- Consider the trajectory of a three-dimensional continuous time dynamical system:
 - $\frac{d\mathbf{x}}{dt} = F(\mathbf{x}, p)$
 - $\mathbf{x}$ is the three-dimensional vector
 - p is the parameter that can vary between some range $p^* > p > -p^*$
- Consider a surface of section and measure every time the dynamical trajectory pierces the surface.
 - Accumulate a time series of data from the trajectories and approximate $F(\mathbf{x}, p)$
 - Linearize around a fixed point
 - Linear state feedback law

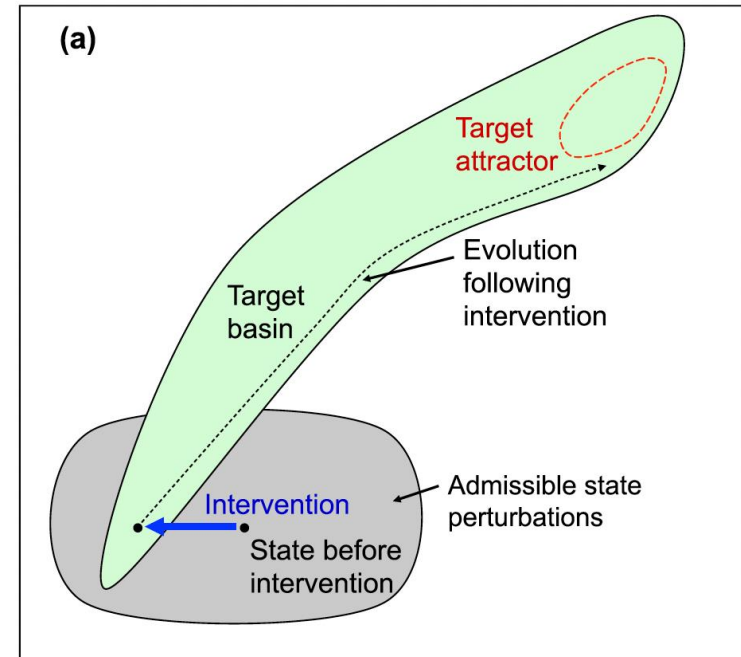
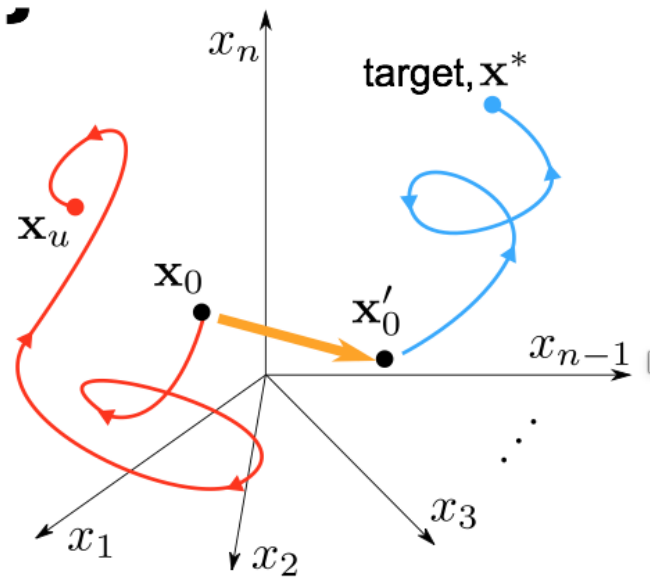


(Image Y.-Y. Liu)

Non-linear dynamics – basin structure key

“Kicking control”

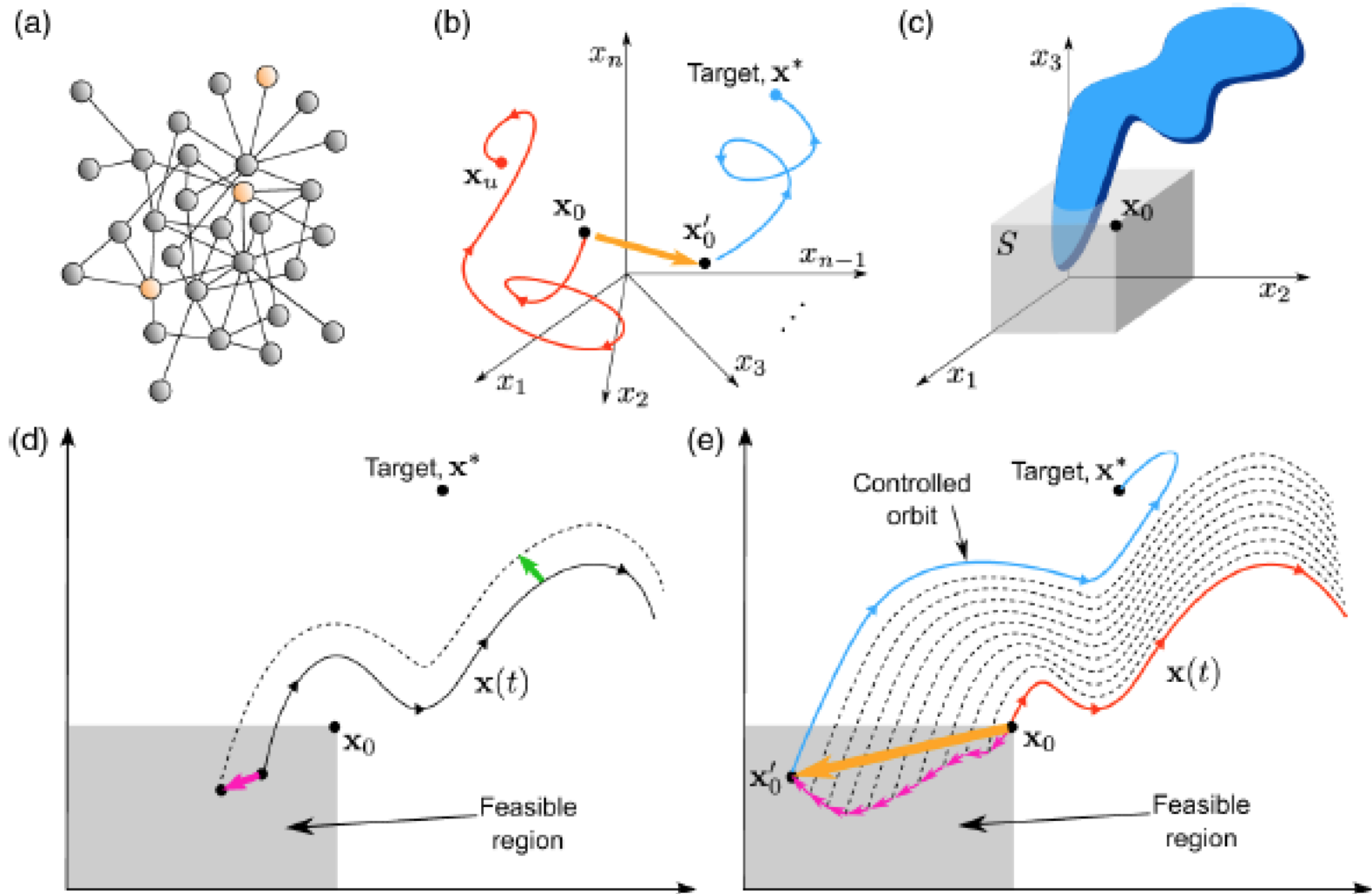
Exploit basins of attraction and natural phase-space trajectories



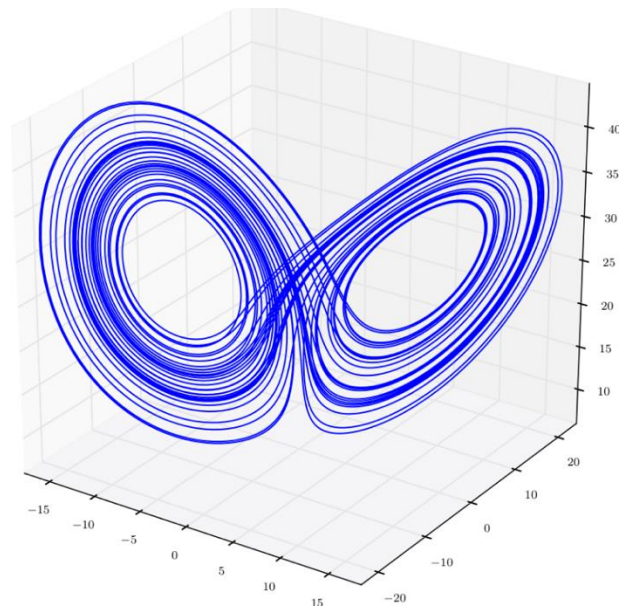
S P Cornelius, WL Kath, and AE Motter. “Realistic control of network dynamics”. *Nature Communications*, 4, 2013.

AE Motter. “Networkcontrology”. *Chaos*, 25, 2015.

S P Cornelius, WL Kath, and AE Motter. "Realistic control of network dynamics". *Nature Communications*, 4, 2013.



Differential equations only work for low dimensional systems.



Data driven model discovery

- System identification** / network inference

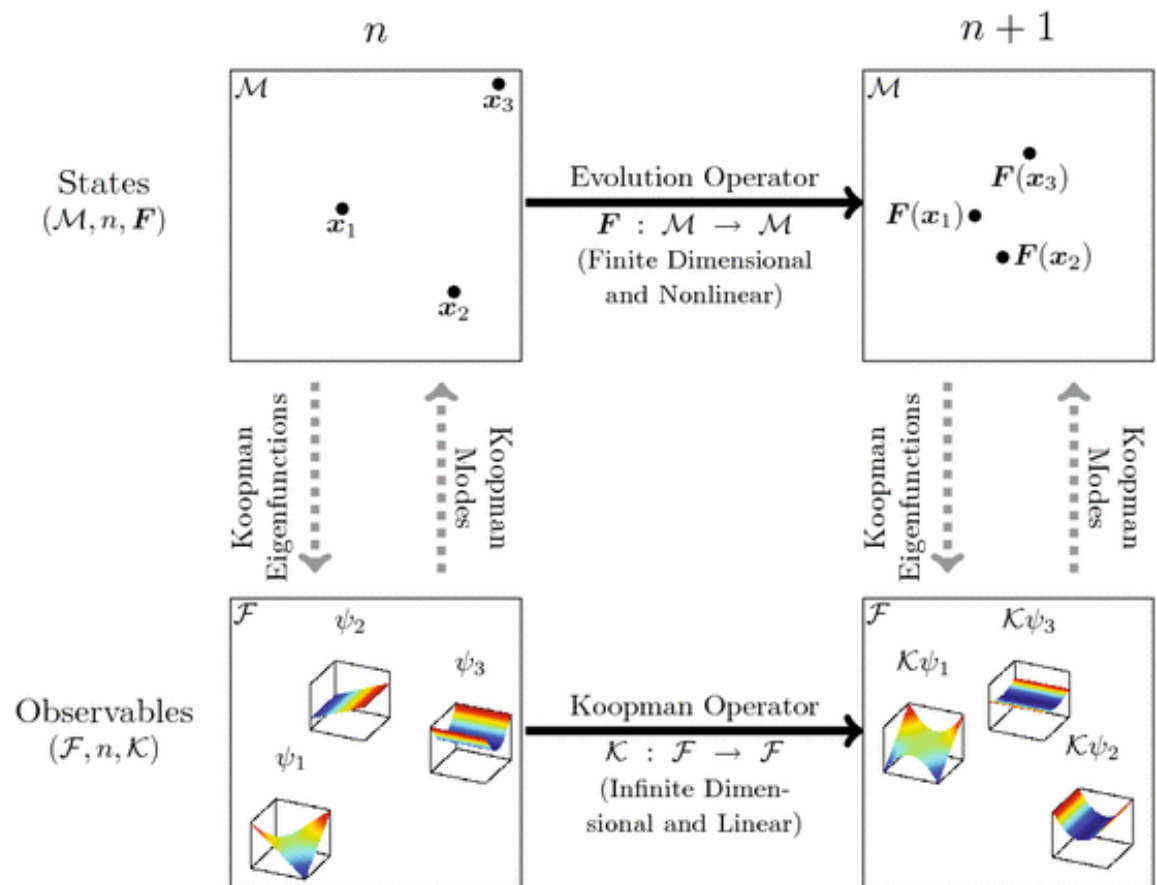
- Discover effective equations of motion from time series data

$$\phi(t+1) \approx \hat{A}\phi(t) \quad \text{where} \quad \hat{A} = \arg \min_{A \in \mathbb{R}^{|E| \times |E|}} \|\Phi_0 - A\Phi_1\|$$

- Operator theory**

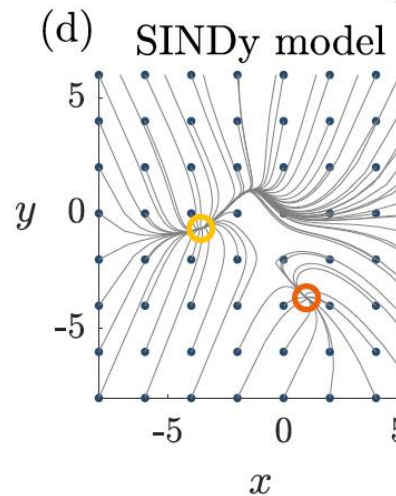
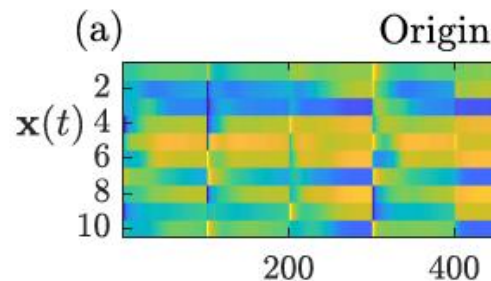
(e.g. Koopman operators)

- Map a nonlinear dynamics to an infinite dimensional, linear dynamics
- Identify the key modes of interest

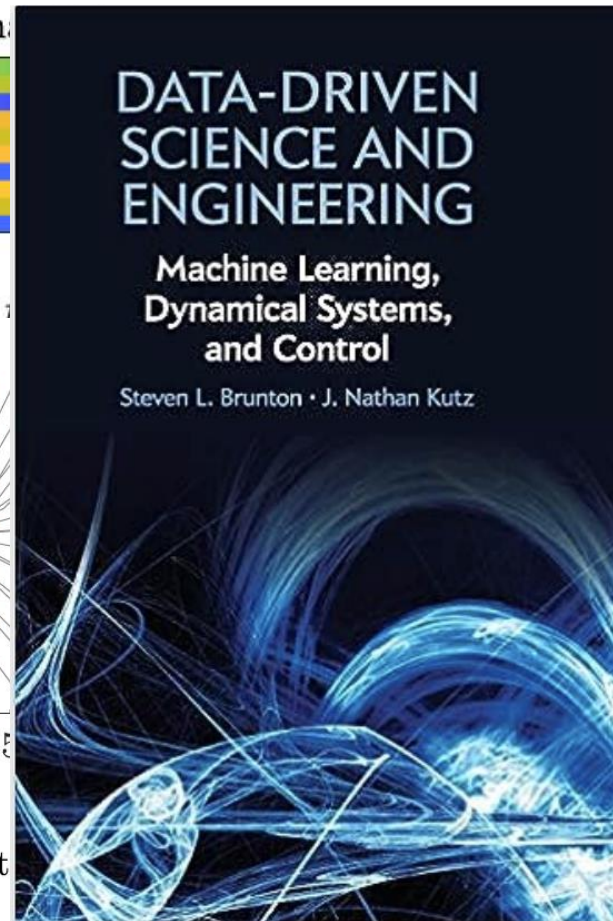


Data driven model discovery for control

- Find a **sparse (low dimensional) representation**, e.g., using SINDy*
- Map out **fixed points** in the low-dimensional space and develop strategic interventions



- stable fixed point
- initial values
- trajectories

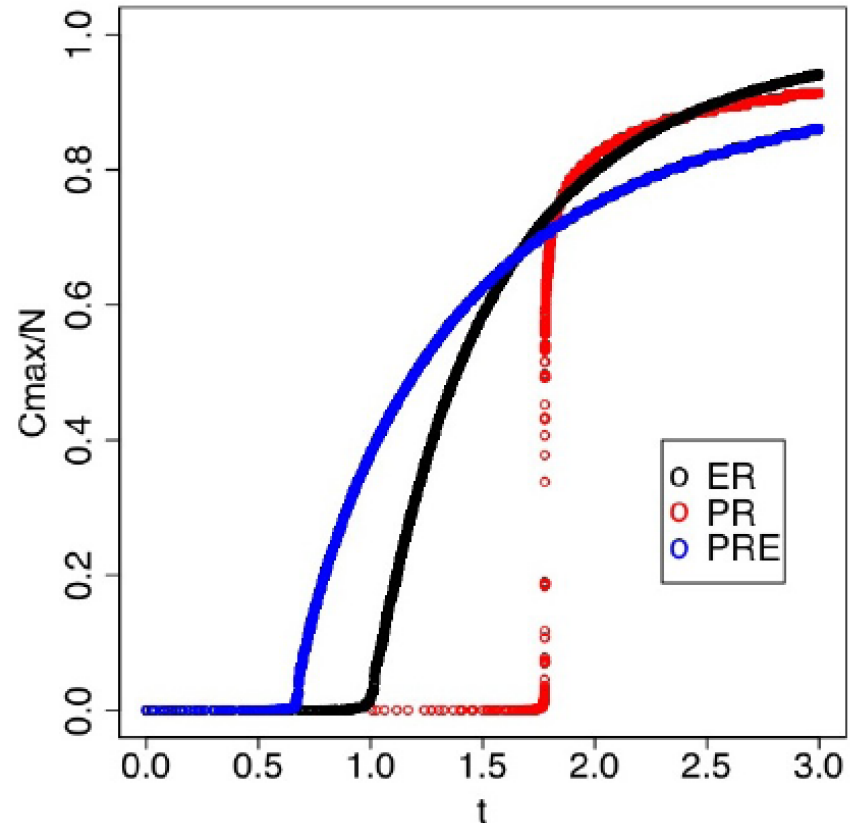


2019, Cambridge Univ. Press

Control of phase transitions

Design small interventions that enhance or delay the onset of phase transitions in a complex network.

- **Enhance** – similar to **ER** but with earlier onset.
- **Delay** – Extremely abrupt



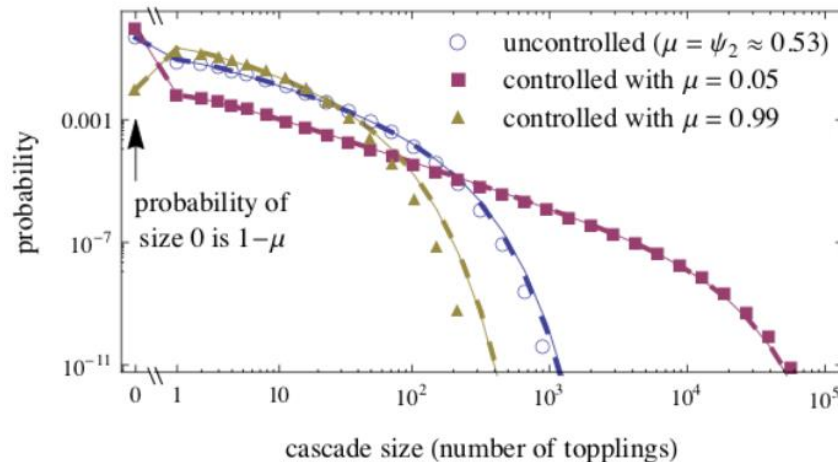
“Explosive percolation in random networks”, D Achlioptas, RM D’Souza, J Spencer, *Science* 323 (5920), 1453-1455, 2009.

Control of SOC

Controlling the BTW model away from the SOC state

Noël, Brummitt, R.D., Phys. Rev. Lett. 111 0780701, 2013

Control parameter μ :
probability grain lands on a node at threshold*



- Avoid cascades,
 $\mu = 0.05 \rightarrow$ larger
cascades when they do
occur.

- Ignite cascades,
 $\mu = 0.99 \rightarrow$ smaller
cascades, but more
frequent.

- Tradeoffs and timescales!

- Ignite cascades, no large cascades but also no profit
- Suppress cascades, no failure for a long time, but massive when happens

III. Social Systems

Modeling Influence & Opinion Dynamics



Social networks and control interventions

Mathematical models of social behavior

Analyze extent of epidemic spreading, product adoption, etc:

- Thresholds models
- Voter models
- Opinion dynamics
(e.g. The Naming game)
- Percolation
- Game theory
- Cascades

INSIDE SCIENCE NEWS SERVICE

Zealots Help Sway Popular Opinions



Image credit: Gabriel Saldana via Flickr | <http://bit.ly/1E9tJCQ>

Rights information: <http://bit.ly/1dWcOPS>

Enthusiasts can greatly influence the adoption of new ideas.

Originally published: Feb 19 2015 - 10:45am

By: Ker Than, Contributor

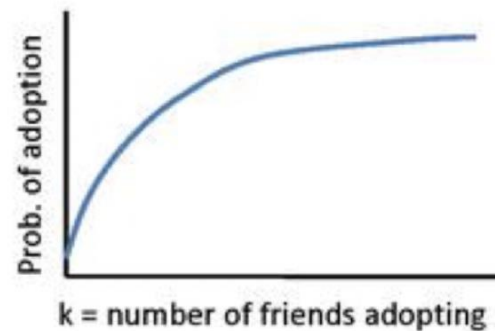
A. Waagen, G. Verma, K. Chan, A. Swami, R. D. *PRE*, 2015.

What mechanism makes an individual change their mind?

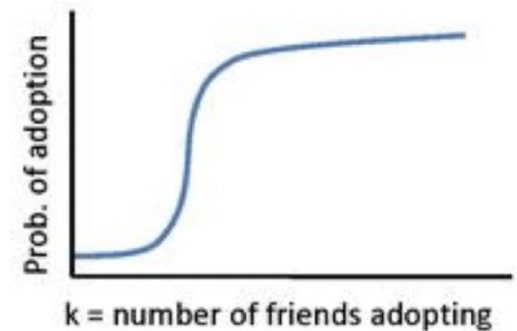
Basic question: What mechanism drives opinion dynamics?

Many competing hypothesis!

- Diminishing returns vs threshold models?



Diminishing returns?



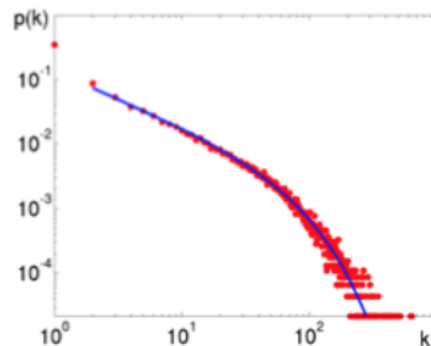
Critical mass?

- Simple contagion versus complex contagion?
- Role of influence and attention? (cusp catastrophes)
- Perhaps people cannot be described by simple equations?

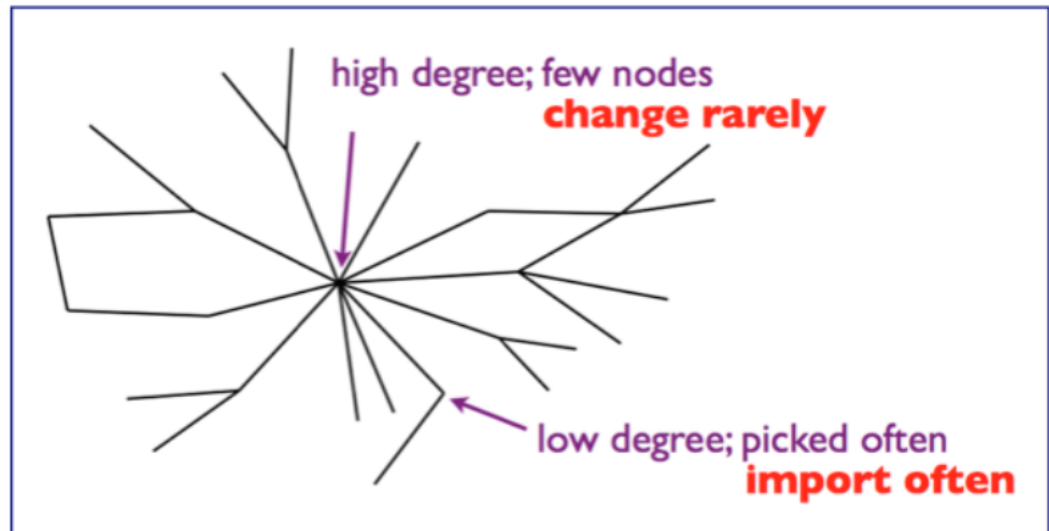
II: The Voter model, “Tell me what to think”

V. Sood, S. Redner, *Phys. Rev. Lett.* 94, 2005.

- At each time step in the process, pick a node at random.
- That node picks a random neighbor, and adopts the opinion of the neighbor.
- Ultimately, only one opinion prevails. The high degree nodes (hubs) win.



Degree distribution

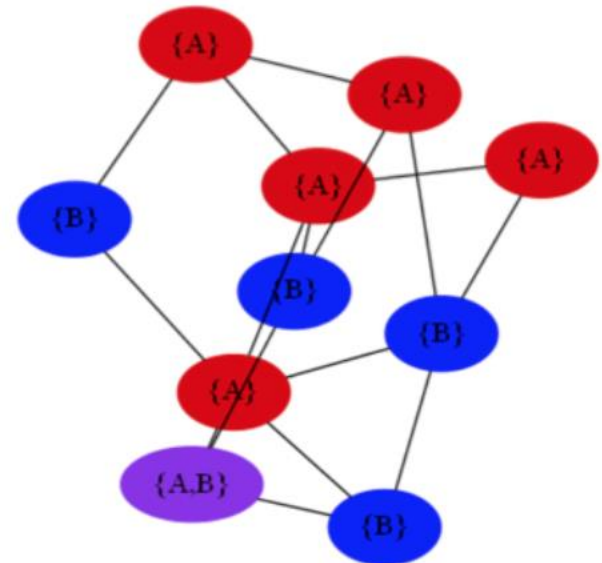
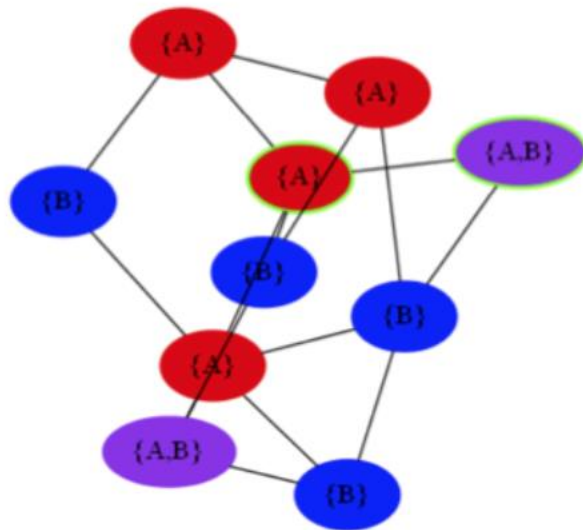


- *Invasion percolation* (the “bully” model) yields the opposite: leaf nodes propagate opinions.

III: “The Naming Game” / open minded individuals

Steels, *Art. Life* 1995; Barrat *et al.*, *Chaos* 2007; Baronchelli *et al.*, *Int. J. Mod. Phys.* 2008.

- Originally introduced for linguistic convergence. Two opinions, A and B.
- And each individual can hold A , B , or $\{A, B\}$.
- Exchange opinions with neighbors and update



The impact of Zealots

Committed individuals who will never change opinions

$$x' = z \left(x + z + \frac{p}{2} \right) - y(x - p),$$

$$y' = z \left(y + z + \frac{q}{2} \right) - x(y - q),$$

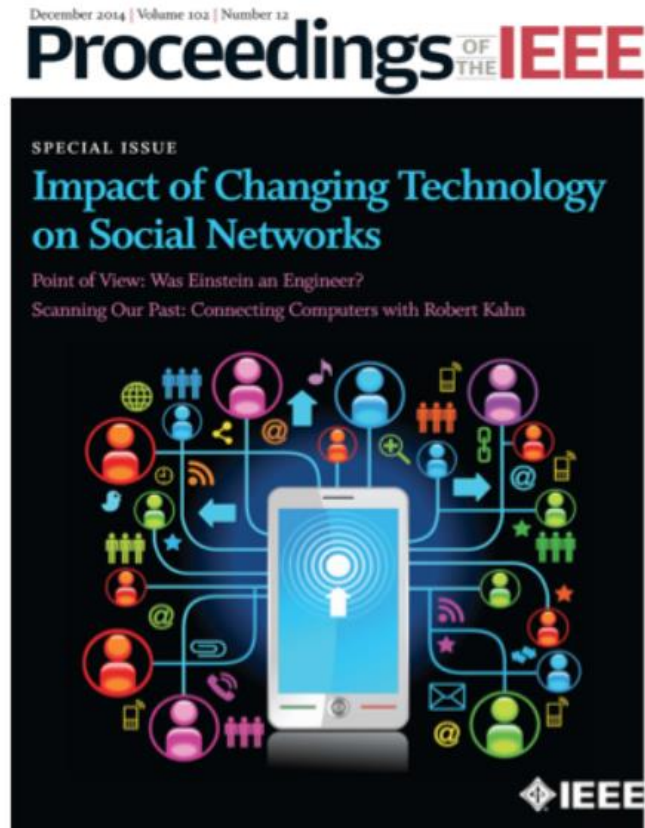
$$z = 1 - x - y.$$

- p is fraction of A zealots; q is fraction of B zealots.
- **Voter model:** A finite **number** of zealots can sway the outcome.
- **Naming game:** A small **fraction** of zealots can sway the outcome.
- **Naming game with multiple choices, k :**
Operating systems; cell phones; political parties; etc
 - Zealots of only one kind: Quickly obey the zealot.
 - Equal fractions of zealots of all kinds: Quickly reach stalemate.

Collective phenomena in social networks

How the online world is changing the game

J. Flack, R.D., editors, PIEEE (2014)



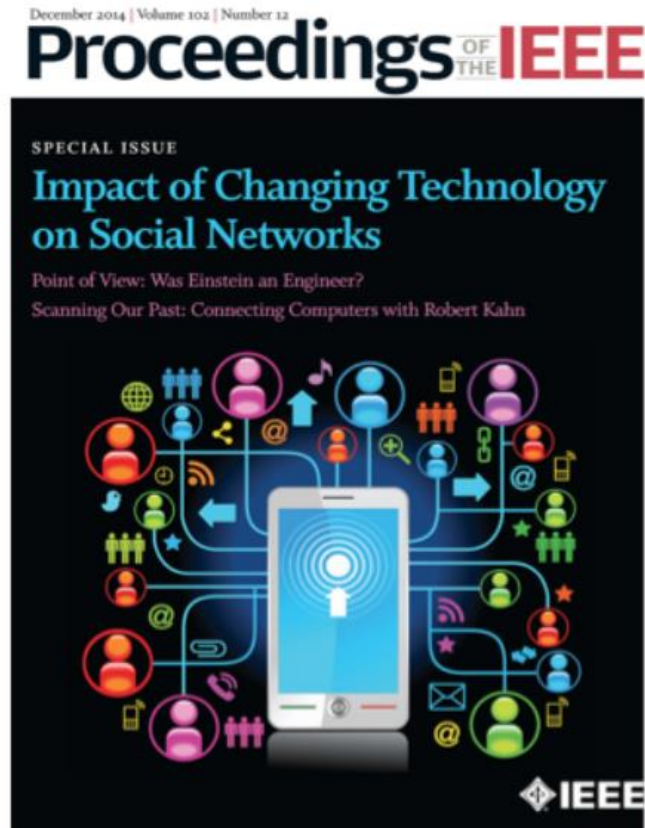
Past: Small, geographically localized social networks, concentrated power and influence

Present: Digital footprint, massive online experimentation, global information, rapid rate of change.

Collective phenomena in social networks

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Past: Small, geographically localized social networks, concentrated power and influence

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Opportunities

Feedback into physics; statistical treatments in control

- Increase network complexity – hypergraphs and multiplex networks
- Driven, far-from-equilibrium models in statistical physics (SOC, KPZ, ASEP)
- Activity driven temporal networks
- ML/AI and data driven model discovery
- Tradeoffs and timescales
- Hybrid treatments (continuous parts/discrete parts; continuafication...)
- Identify a set of paradigmatic problems or benchmark case studies that could be used to validate and contrast different approaches to control complex systems

Commonalities

Control of complex networks across genres

**What to influence and how to influence
and when to influence?**

Commonalities

- **Do we want to control the global behavior or behavior of every single node?**

e.g. Large-scale connectivity in a network vs eliminating smoking in a particular individual

Commonalities

- **What is the minimal amount of knowledge of the system we need in order to control it?**

Learn more:

nature reviews physics

<https://doi.org/10.1038/s42254-023-00566-3>

Perspective

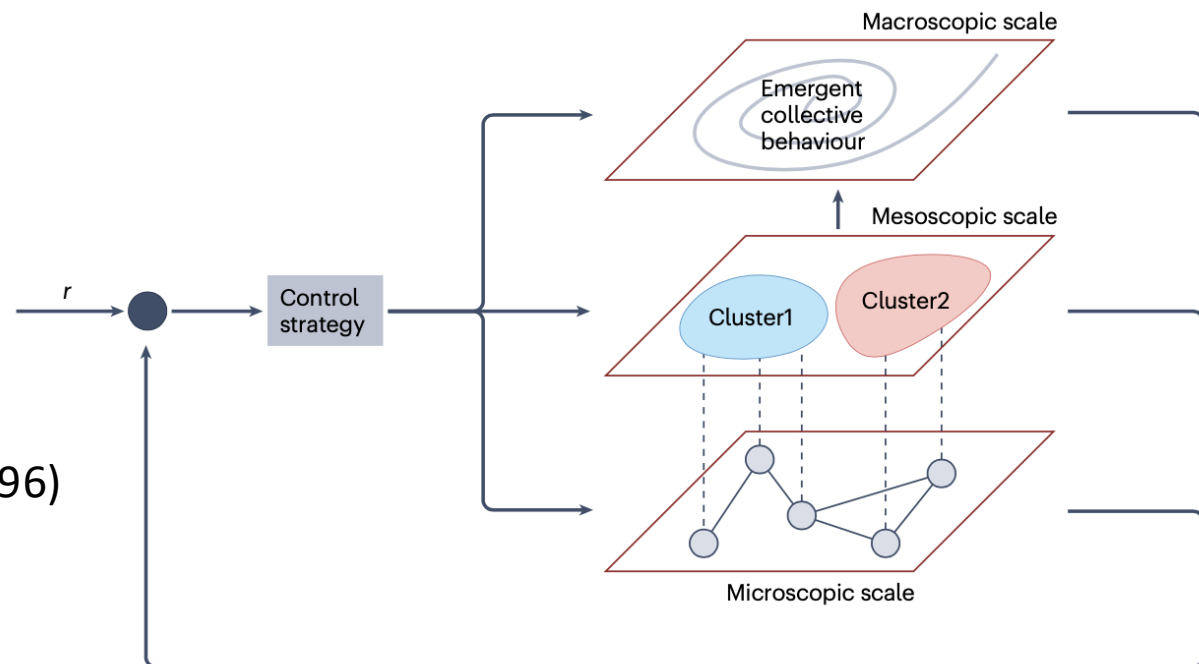
 Check for updates

Controlling complex networks with complex nodes

Published online: 24 March 2023

Raissa M. D'Souza ^{1,2,3} , Mario di Bernardo ^{4,5}  & Yang-Yu Liu ^{6,7} 

(Back to Lecture 10, page 96)



Conclusions

Modern systems are made of interconnected complex systems:
Socio-technical, cyber-physical, eco-social.

Can't necessarily reduce to a set of interconnected differential equations.

➤ Emergent interactions:

- Decoupled states – nodal dynamics & network structure
- BTW meets Kuramoto – sandpile cascades on oscillator networks
- SOC and Dragon kings – self-amplifying cascades

➤ Controlling complex networks:

- Structural control (linear nodal dynamics)
- Non-linear dynamics and basin structure
- Data driven model prediction
- Control of phase transitions and SOC

Thank you!



Guram Mikaberidze



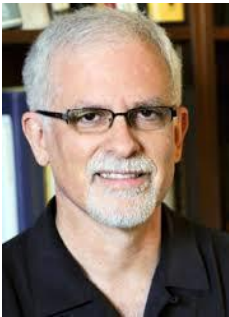
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