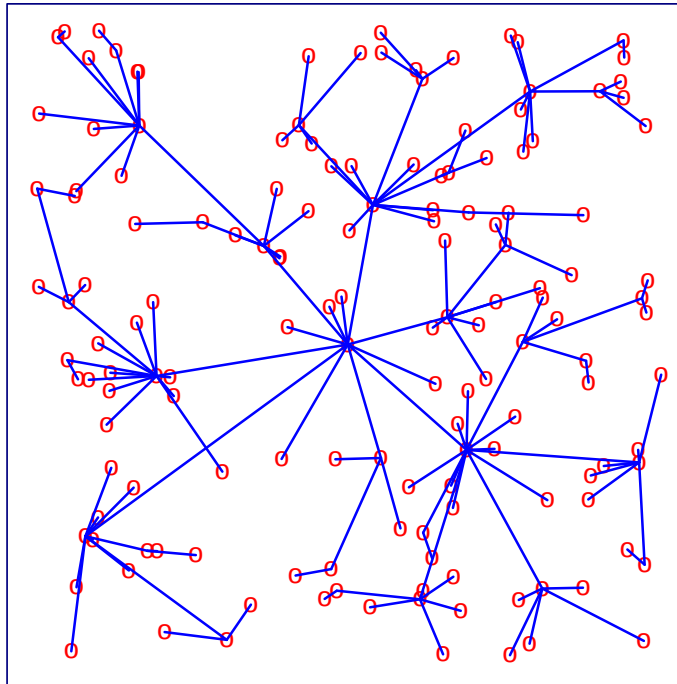


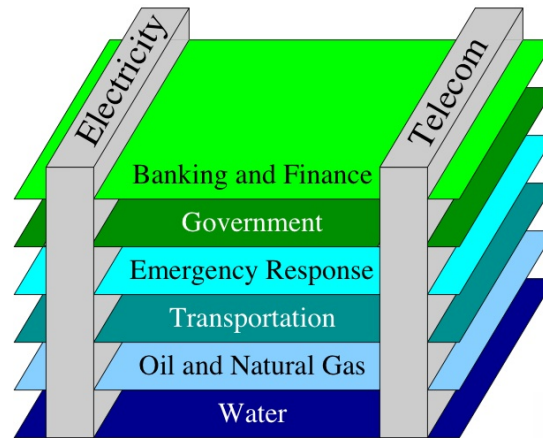
ECS 253 / MAE 253, Lecture 2

April 2, 2025

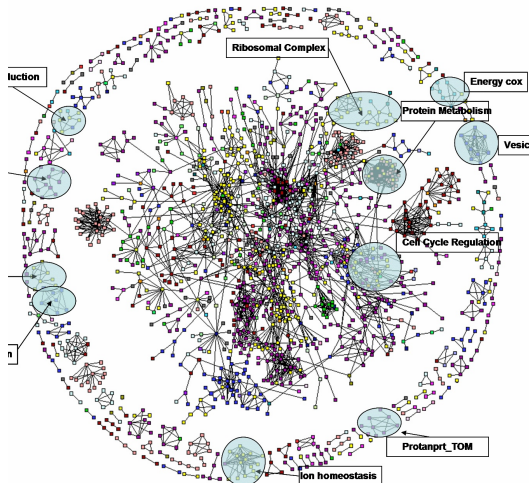


“Power laws, Random graphs, phase transitions”

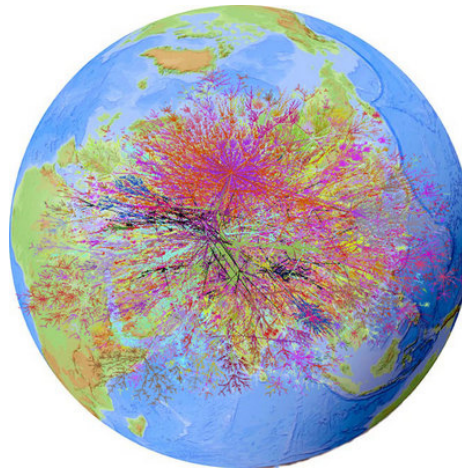
Complex networks are ubiquitous:



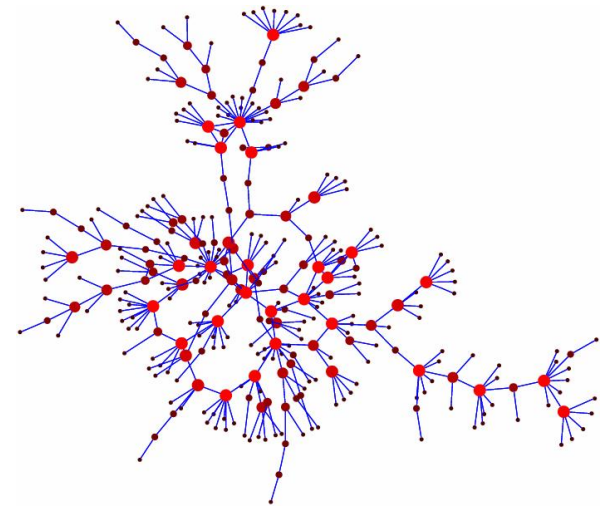
Critical Infrastructure



***Biological & Ecological
networks***



***Information and Communication
technology***



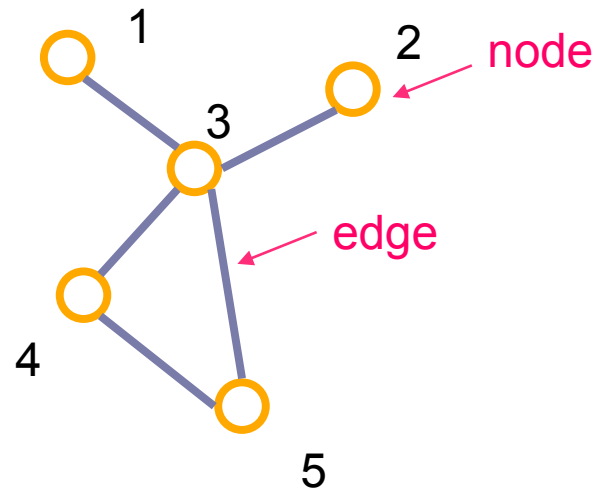
***Social networks:
Economics & Epidemics***

Networks: Physical, Biological, Social, Technological

- **Geometric** versus **virtual** (Internet versus WWW).
- **Natural** /spontaneously arising versus **engineered** /built.
- **Directed** versus **undirected** edges.
- Each network may **optimize** something unique.
- Identifying **similarities** and fundamental **differences** can guide future design/understanding.
- Interplay of **topology** and **function** ?
- Unifying features: – **Broad heterogeneity in node degree.**
– **Small Worlds** (Diameter $\sim \log(N)$).

What are networks?

- Networks are collections of points joined by lines.



“Network” $\equiv$ “Graph”

points	lines	
vertices	edges, arcs	math
nodes	links	computer science
sites	bonds	physics
actors	ties, relations	sociology

Subtle details of edges

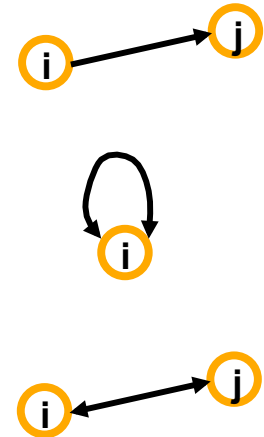
Network elements: edges

- Directed (also called arcs)
 - $A \rightarrow B$ (E_{BA})
 - A likes B, A gave a gift to B, A is B's child
- Undirected
 - $A \leftrightarrow B$ or $A - B$
 - A and B like each other
 - A and B are siblings
 - A and B are co-authors
- Edge attributes
 - weight (e.g. frequency of communication)
 - ranking (best friend, second best friend...)
 - type (friend, relative, co-worker)
 - properties depending on the structure of the rest of the graph:
e.g. betweenness
- Multiedge: multiple edges between two pair of nodes
- Self-edge: from a node to itself

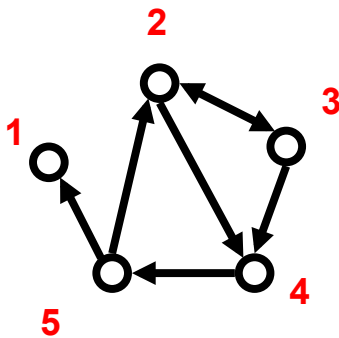
Adjacency matrices

■ Representing edges (who is adjacent to whom) as a matrix

- $A_{ij} = 1$ if node i has an edge to node j
= 0 if node i does not have an edge to j
- $A_{ii} = 0$ unless the network has self-loops
 - If self-loop, $A_{ii}=1$
- $A_{ij} = A_{ji}$ if the network is undirected, or if i and j share a reciprocated edge



Example:

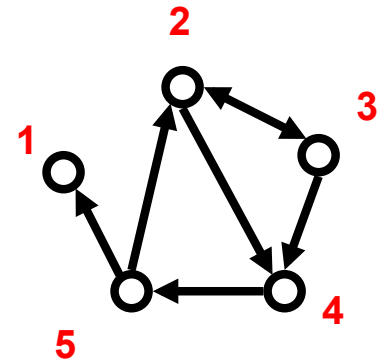


$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Adjacency lists

■ Edge list

- 2 3
- 2 4
- 3 2
- 3 4
- 4 5
- 5 2
- 5 1



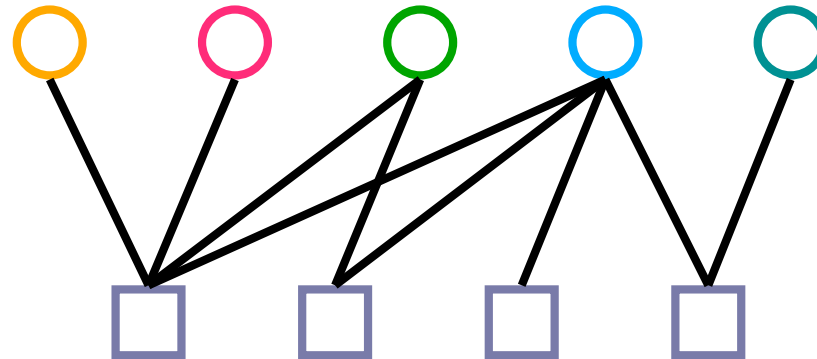
■ Adjacency list

- is easier to work with if network is
 - large
 - sparse
- quickly retrieve all neighbors for a node
 - 1:
 - 2: 3 4
 - 3: 2 4
 - 4: 5
 - 5: 1 2

Beyond simple networks:

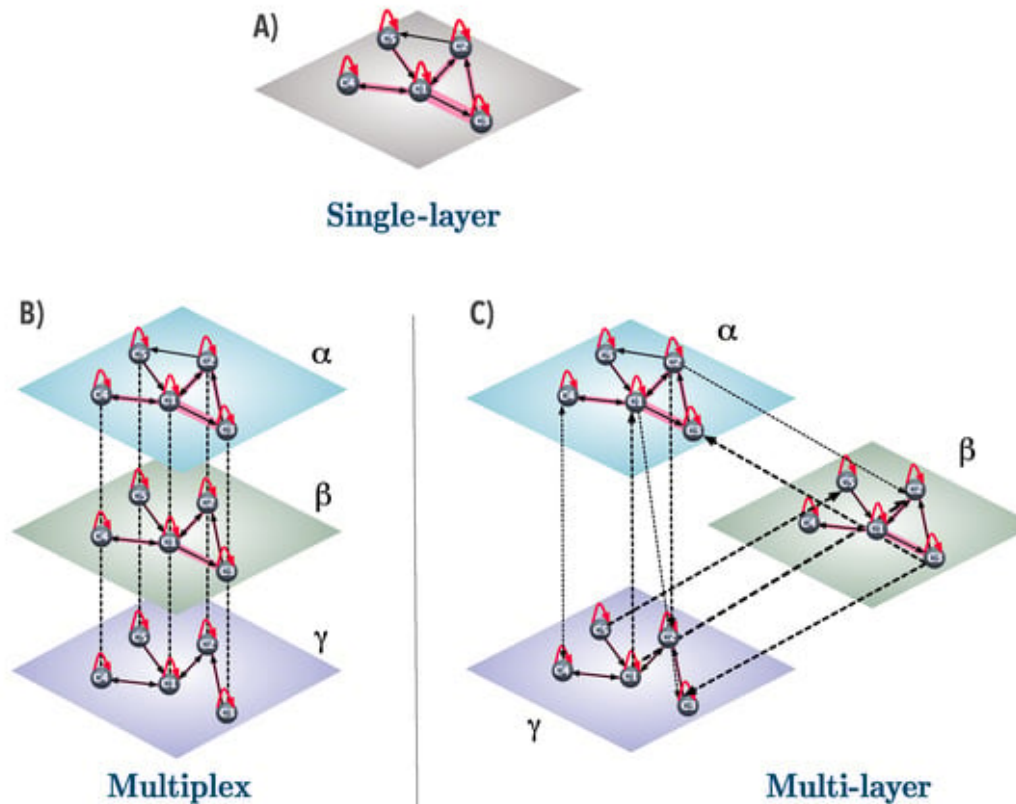
Bipartite (two-mode) networks

- edges occur only between two groups of nodes, not within those groups
- for example, we may have individuals and *events*
 - directors and boards of directors
 - customers and the items they purchase
 - metabolites and the reactions they participate in



Beyond simple networks: Multiplex and multi-layered

Image from Alves et al, Entropy, 2018



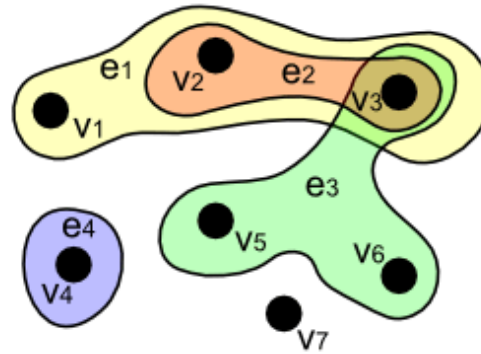
- B) **Multiplex:** the same set of nodes have multiple types of relationships, each one described by a layer.
- C) **Multi-layer:** The nodes in each layer can be distinct.

Beyond simple networks:

HyperGraphs

- Edges join more than two nodes at a time (*hyperEdge*)

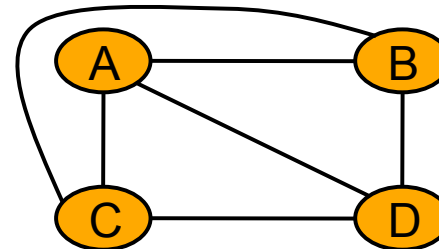
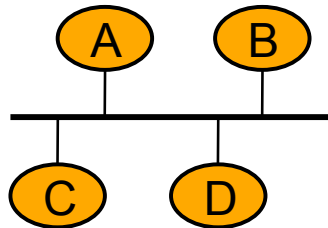
- Affiliation networks



- Examples

- Families
- Subnetworks

- Chemical reactions
- co-author networks

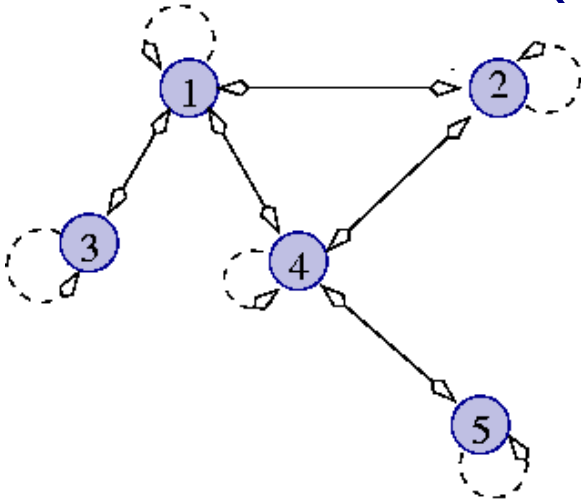


Can be transformed to a *bipartite network*

NETWORK TOPOLOGY; simple edges

Binary connectivity matrix, M :

$$M_{ij} = \begin{cases} 1 & \text{if edge exists between } i \text{ and } j \\ 0 & \text{otherwise.} \end{cases}$$



$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} = M$$

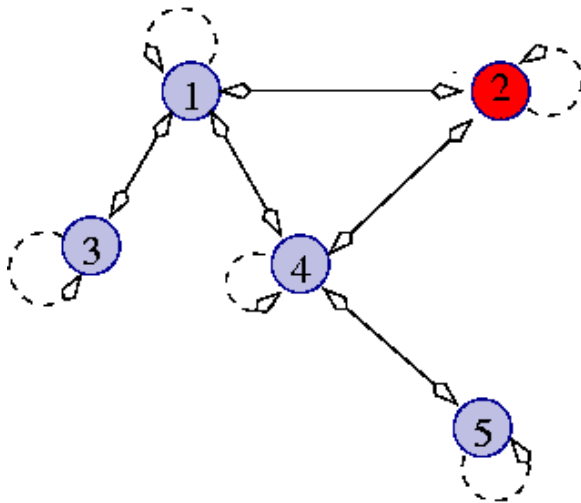
Node **degree** is number of links.

Network Activity: FLOWS on NETWORKS

(Spread of disease, routing data, materials transport/flow, gossip spread/marketing)

FLows on NETWORKS : Random walks

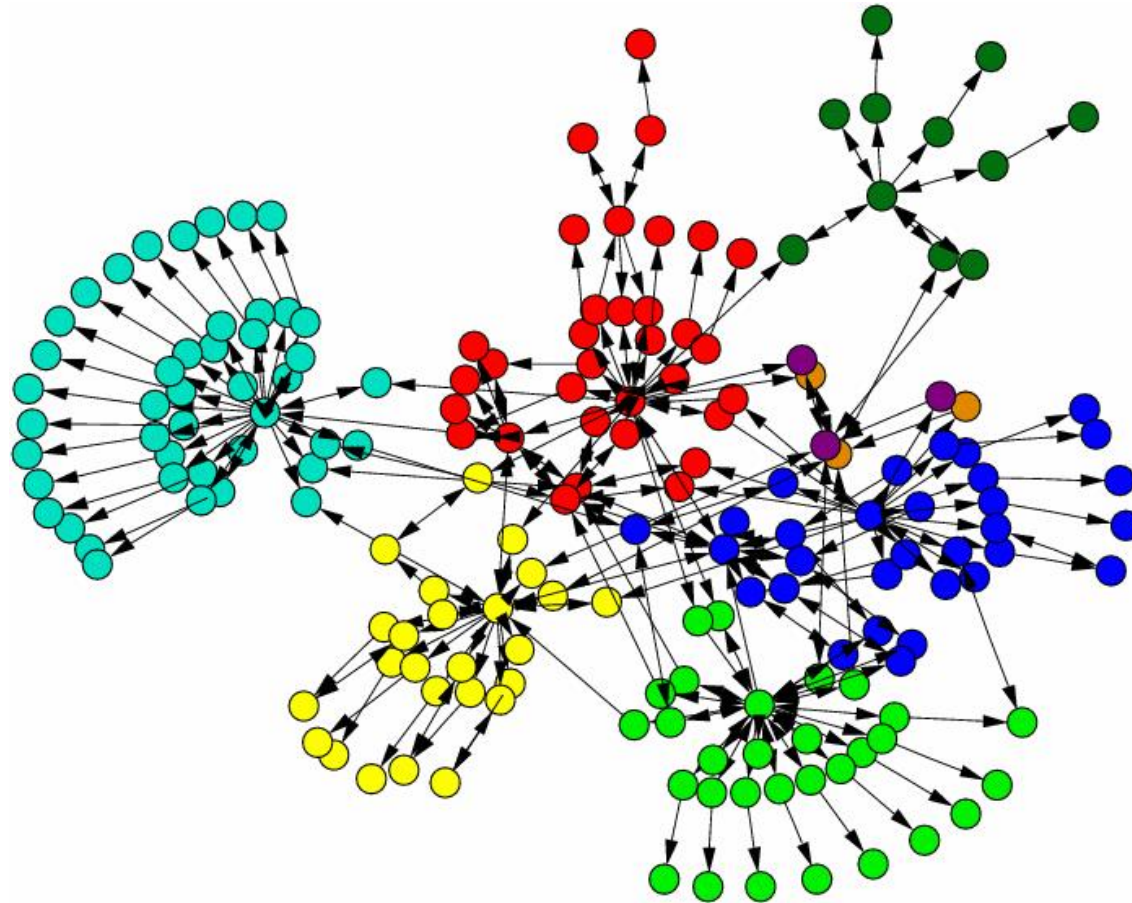
Random walk on the network has state transition matrix, P :
(Column normalize the adjacency matrix)



$$\begin{pmatrix} 1/4 & 1/3 & 1/2 & 1/4 & 0 \\ 1/4 & 1/3 & 0 & 1/4 & 0 \\ 1/4 & 0 & 1/2 & 0 & 0 \\ 1/4 & 1/3 & 0 & 1/4 & 1/2 \\ 0 & 0 & 0 & 1/4 & 1/2 \end{pmatrix} = P$$

The eigenvalues and eigenvectors convey much information.
Markov Chains, Spectral Gap.

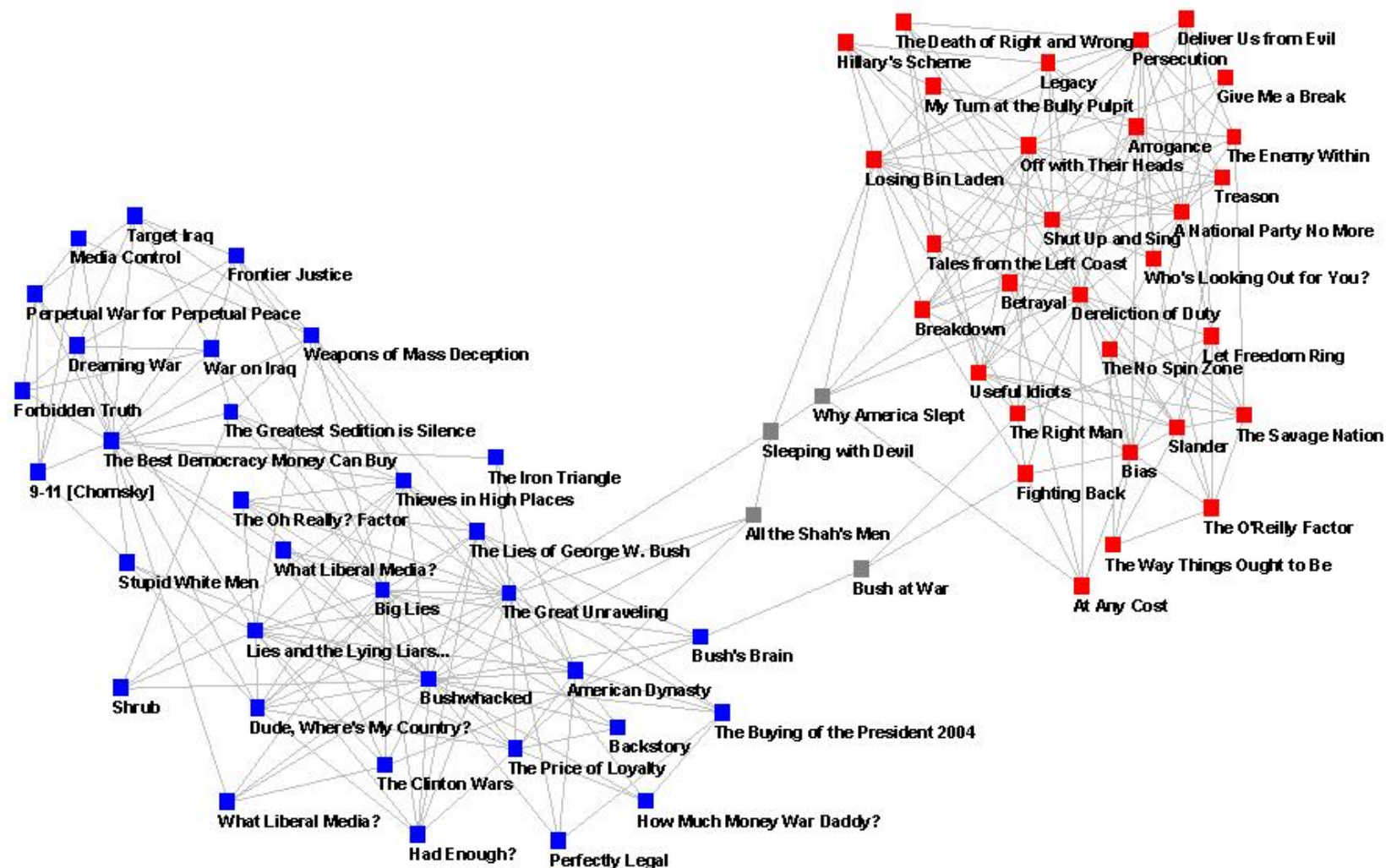
Random walk on the WWW is the “Page Rank”



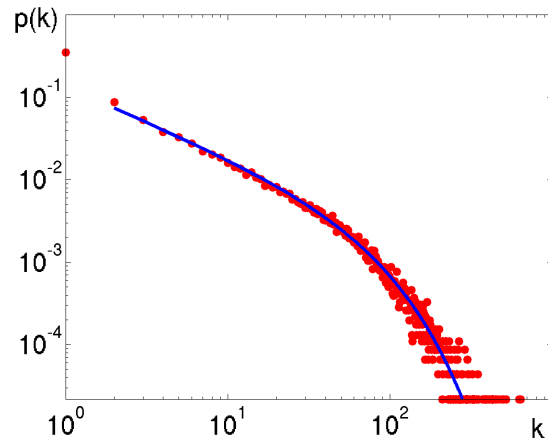
Page Rank of a node is the steady-state random walk occupancy probability.

(We will discuss building a search engine in detail later.)

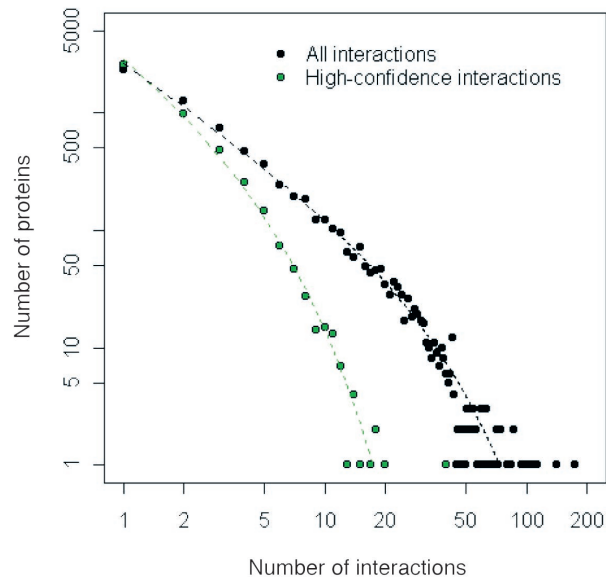
Example Eigen-technique: Community structure (Political Books 2004)



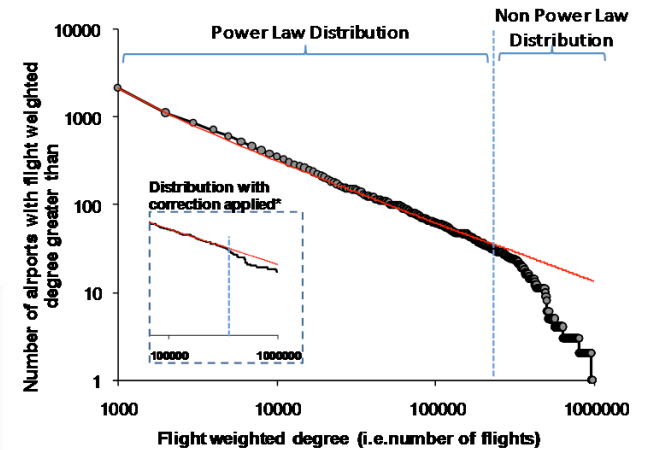
Back to topology: Broad scale degree distributions



Social contacts
Szendrői and Csányi



Protein interactions
Giot et al Science 2003



Airport traffic
Bounova 2009

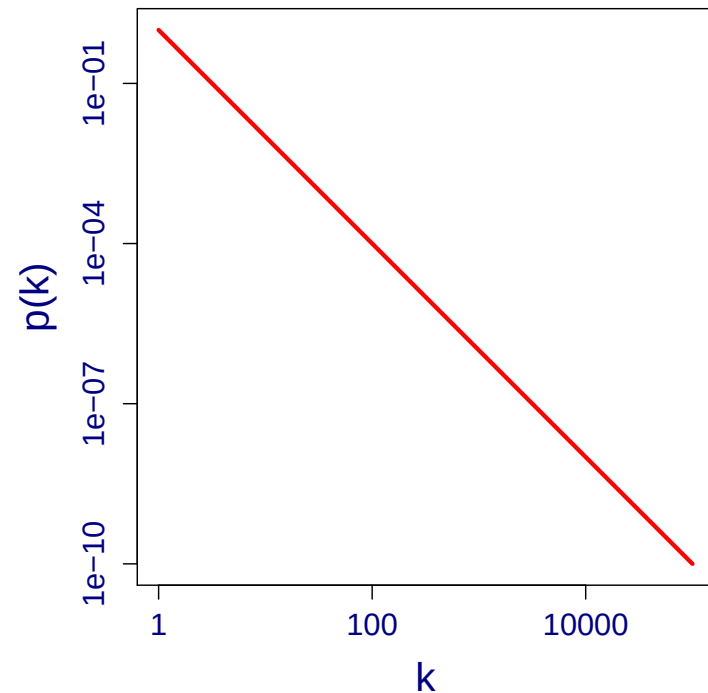
- A few hubs, dominated by leaves
- Small data sets, power laws vs log normal, stretched-exponential, etc...
 - Exceptions: Power grids? Router-level Internet?

Degree distribution

- Often observe “**heavy-tailed**” / “**broad-scale**” degree distributions.
- The simplest example of such a distribution is a power law (Pareto distribution).

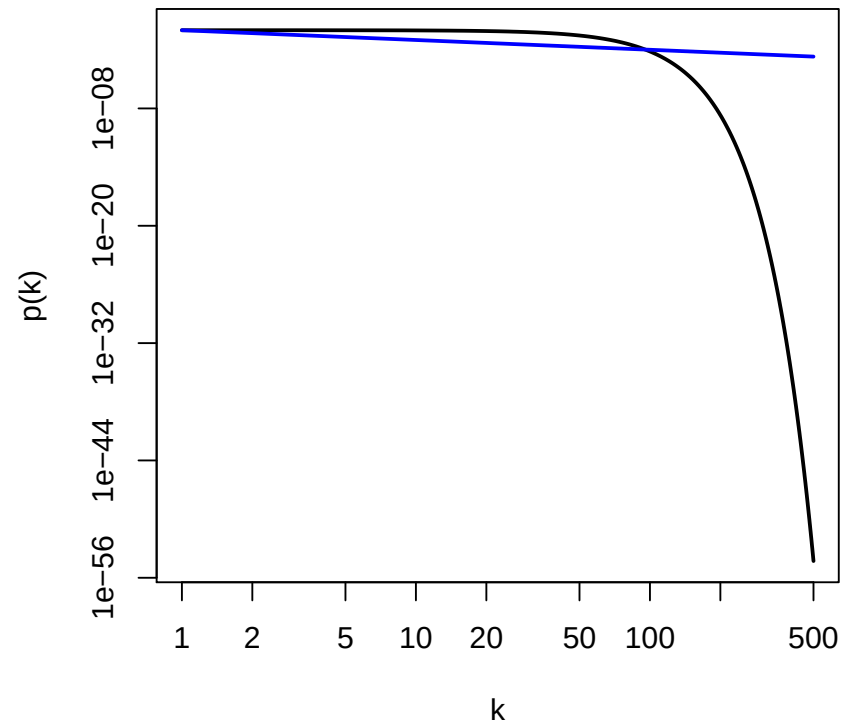
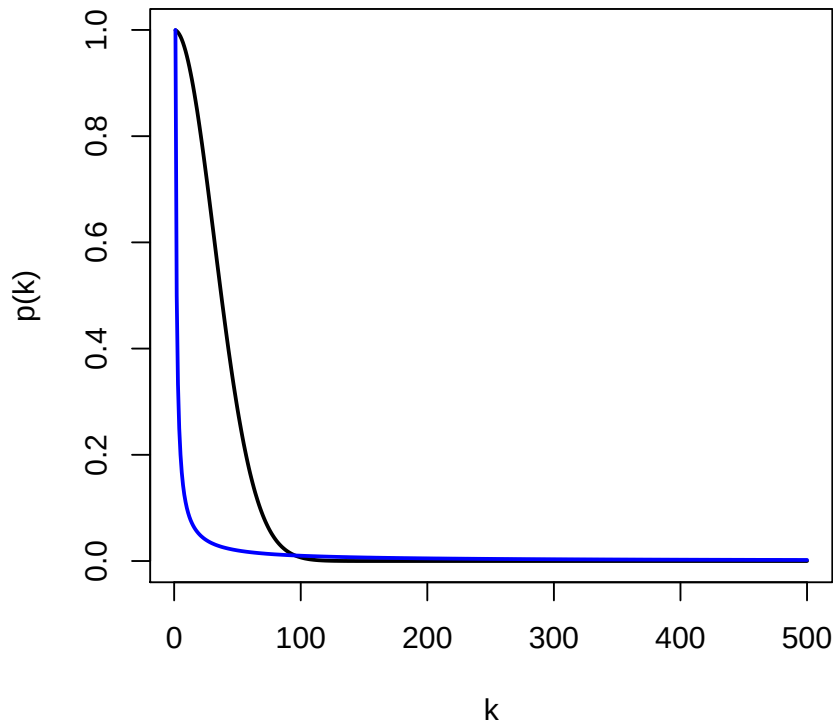
$$p_k \sim k^{-\gamma}$$

$$\ln p_k \sim -\gamma \ln k$$



Power Laws versus Bell Curves: “Heavy tails”

- Power law distribution: $p_k \sim k^{-\gamma}$.
- Gaussian distribution: $p_k \sim \exp(-k^2/2\sigma^2)$.



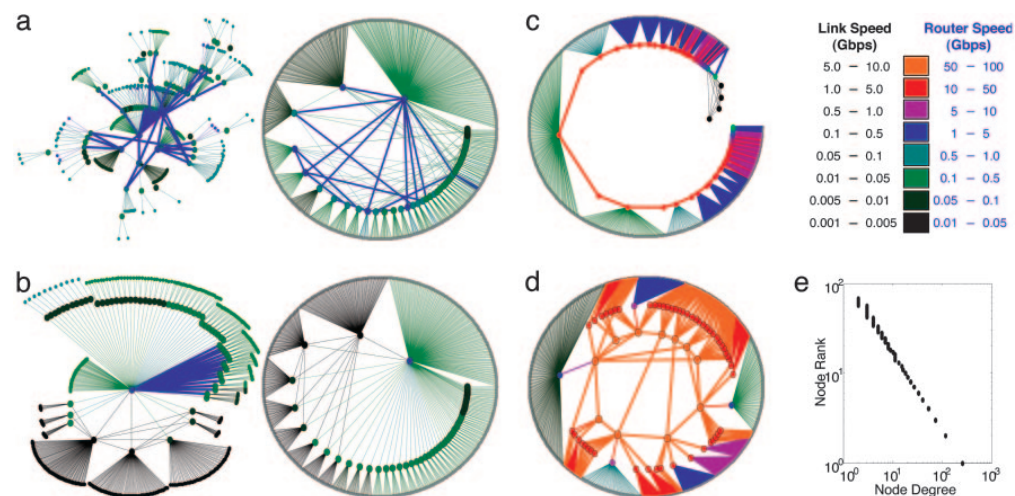
- Most nodes have low degree
- But a few nodes are hubs, with massive degree

Many network growth models produce power law degree distribution (we will study some of these)

- Preferential attachment
- Copying models (WWW, biological networks, ...)
- Optimization models

Degree distribution misses other structure.

- Doyle, et. al.,
PNAS **102** (4)2005.



Power law probability distributions: $p_k = Ak^{-\gamma}$ with $\gamma > 0$

- $0 \leq p_k \leq 1 \quad \forall k$ which are valid degrees (typically $k \in \mathbb{Z}^+$).
- Must be properly normalized:

$$\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} \frac{A}{k^{\gamma}} = 1$$

- Approximating discrete sum by integral:

$$\begin{aligned} 1 &= \int_{k=1}^{\infty} \frac{A}{k^{\gamma}} = - \left(\frac{A}{\gamma - 1} \right) \frac{1}{k^{(\gamma-1)}} \Big|_{k=1}^{\infty} \\ &= \left(\frac{A}{1 - \gamma} \right) \left(\frac{1}{\infty^{(\gamma-1)}} - 1 \right) \end{aligned}$$

- Finite requirement means $\gamma > 1$, in which case $A = (\gamma - 1)$.

The first moment (the mean)

Recall, $p_k = \frac{A}{k^\gamma}$

- Mean degree:

$$\langle k \rangle = \sum_{k=1}^{\infty} k p_k \approx \int_{k=1}^{\infty} k p_k dk = \int_{k=1}^{\infty} \frac{A}{k^{(\gamma-1)}} dk$$

Diverges (i.e., $\langle k \rangle \rightarrow \infty$) if $\gamma \leq 2$.

The second moment and the variance

- Second moment:

$$\langle k^2 \rangle = \sum_{k=1}^{\infty} k^2 p_k \approx \int_{k=1}^{\infty} k^2 p_k dk = \int_{k=1}^{\infty} \frac{A}{k^{(\gamma-2)}} dk$$

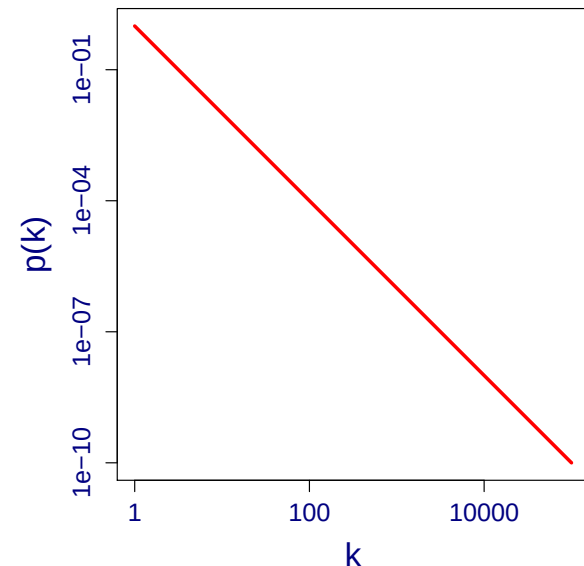
Diverges (i.e., $\langle k^2 \rangle \rightarrow \infty$) if $\gamma \leq 3$.

- Variance = $\langle k^2 \rangle - \langle k \rangle^2$, likewise diverges if $\gamma \leq 3$.

Properties of a power law PDF (Summary)

(PDF = probability density function)

- To be a properly defined probability distribution need $\gamma > 1$.
- For $1 < \gamma \leq 2$, both the average $\langle k \rangle$ and variance σ^2 are infinite!
- For $2 < \gamma \leq 3$, average $\langle k \rangle$ is finite, but variance σ^2 is infinite!
- For $\gamma > 3$, both average and variance finite.



Why a power law is “scale-free”

- Power law for “x”, means “scale-free” in x:

$$p(bx) = (bx)^{-\gamma} = b^{-\gamma}p(x)$$

$$\boxed{\frac{p(bk)}{p(k)} = b^{-\gamma}} \text{ regardless of } k.$$

In contrast consider: $p(k) = A \exp(-k)$.

So $p(bk) = A \exp(-bk)$.

$$\boxed{\frac{p(bk)}{p(k)} = \exp[-k(b - 1)]} \text{ dependent on } k$$

Self-similar/scale-free fractal structures



Sierpinski Sieve/Gasket/Fractal, $N \sim r^d$.

When r doubles, N triples: $3 = 2^d$

$$d = \log N / \log r = \log 3 / \log 2$$

Power laws in the real world

Confusion

- Power law
- Log normal
- Weibull
- Stretched exponential

All of these distributions can look the same! (Especially when we are dealing with finite data sets — not enough data to get good statistics).

How to deal with real data

- Can adjust bin size: increase exponentially with degree.
- Consider the Cumulative PDF (the CDF): $P_k = \sum_{l=k}^{\infty} p_l$.

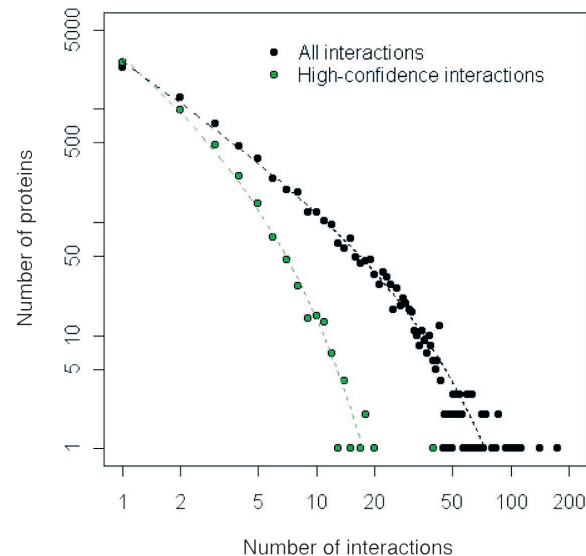
Good reviews:

- Aaron Clauset, Cosma R. Shalizi, M. E. J. Newman. “Power-Law Distributions in Empirical Data”, *SIAM Review*, Vol. 51, No. 4. (2009), pp. 661-703.
- A Brief History of Generative Models for Power Law and Lognormal Distributions Michael Mitzenmacher, *Internet Math.* Vol 1 (2003), 226-251.

Power law with exponential tail

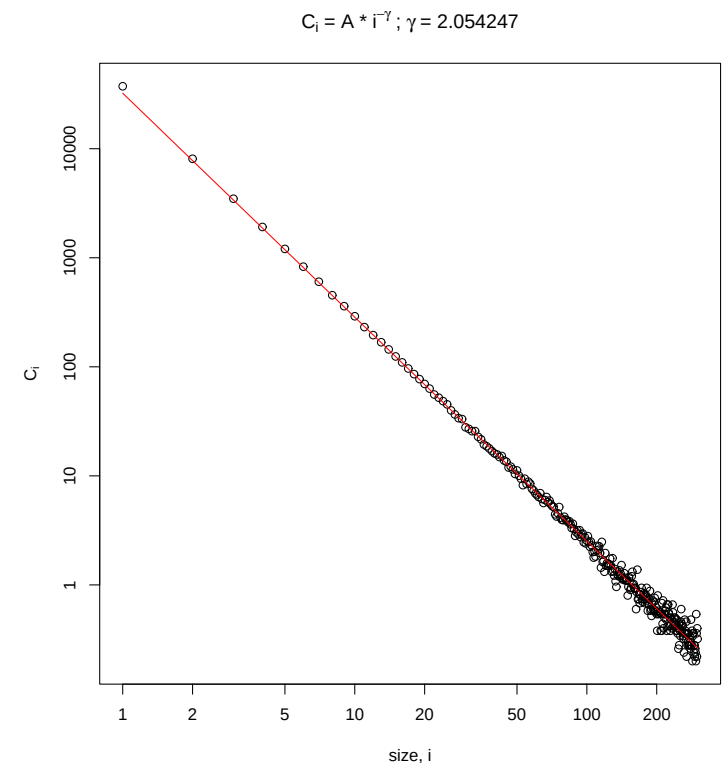
Ubiquitous empirical measurements:

System with: $p(x) \sim x^{-B} \exp(-x/C)$	B	C
Full protein-interaction map of <i>Drosophila</i>	1.20	0.038
High-confidence protein-interaction map of <i>Drosophila</i>	1.26	0.27
Gene-flow/hybridization network of plants as function of spatial distance	0.75	10^5 m
Earthquake magnitude	1.35 - 1.7	$\sim 10^{21}$ Nm
Avalanche size of ferromagnetic materials	1.2 - 1.4	$L^{1.4}$
ArXiv co-author network	1.3	53
MEDLINE co-author network	2.1	~ 5800
PNAS paper citation network	0.49	4.21



True power laws are observed in many systems

- Signature of a system at the “critical point” of a phase transition.
Foundation of renormalization group approach to critical phenomena
- Random graphs at critical point;
component sizes: $N_k \sim k^{-5/2}$
(Note, $\gamma = 2.5$)



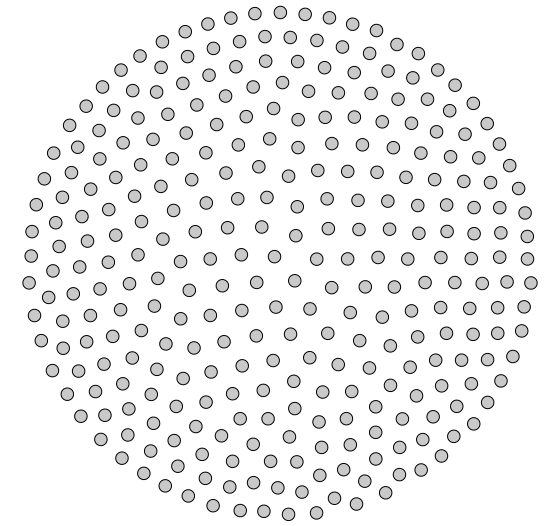
The origins of network theory: Random graphs

What does a “typical” graph with n vertices and m edges look like?

- P. Erdős and A. Rényi, “On random graphs”, *Publ. Math. Debrecen.* **6**, 1959.
- P. Erdős and A. Rényi, “On the evolution of random graphs”, *Publ. Math. Inst. Hungar. Acad. Sci.* **5**, 1960.
- E. N. Gilbert, “Random graphs”, *Annals of Mathematical Statistics* **30**, 1959.

Erdős-Rényi random graphs

- Consider a *labelled* graph. Each vertex has a label ranging from $[1, 2, 3, \dots, n]$, for a set of n vertices. (This will make counting and analysis easier.)



- Let E denote the total number of edges possible:

$$E = \binom{N}{2} = \frac{N!}{2!(N-2)!} = \frac{N(N-1)}{2}$$

(If directed edges, we would not divide by 2).

Two formulations

- 1) $\mathcal{G}(n, p)$: The *ensemble* of graphs constructed by putting in edges with probability p , independent of one another. (An edge is present with probability p and absent with probability $[1 - p]$.)

Let $G(n, p)$ denote a random realization of $\mathcal{G}(n, p)$.

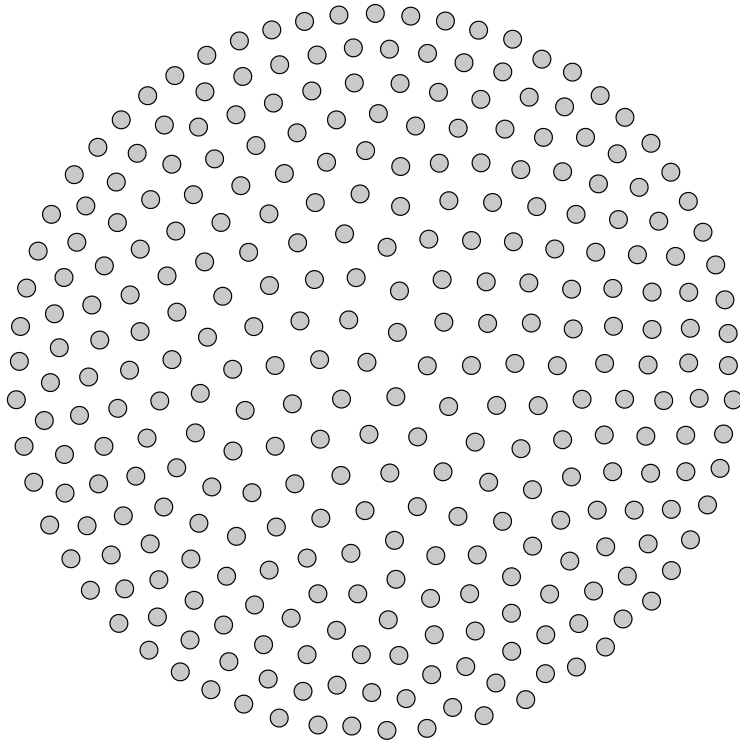
- 2) $\mathcal{G}(n, m)$: The ensemble of all graphs with n nodes and exactly m edges.

Let $G(n, m)$ denote a random realization of $\mathcal{G}(n, m)$.

- The two are almost interchangeable with $m = pE$. (Recall, E is total number of edges possible).
- We will focus on $G(n, p)$.

The “classic” random graph, $G(N, p)$ (The Null Model)

- P. Erdős and A. Rényi, “On random graphs”, *Publ. Math. Debrecen*. 1959.
- P. Erdős and A. Rényi, “On the evolution of random graphs”, *Publ. Math. Inst. Hungar. Acad. Sci.* 1960.
- E. N. Gilbert, “Random graphs”, *Annals of Mathematical Statistics*, 1959.



- Start with N isolated vertices.
- Add random edges one-at-a-time.
 $N(N - 1)/2$ total edges possible.
- After E edges, probability p of any edge is $p = 2E/N(N - 1)$

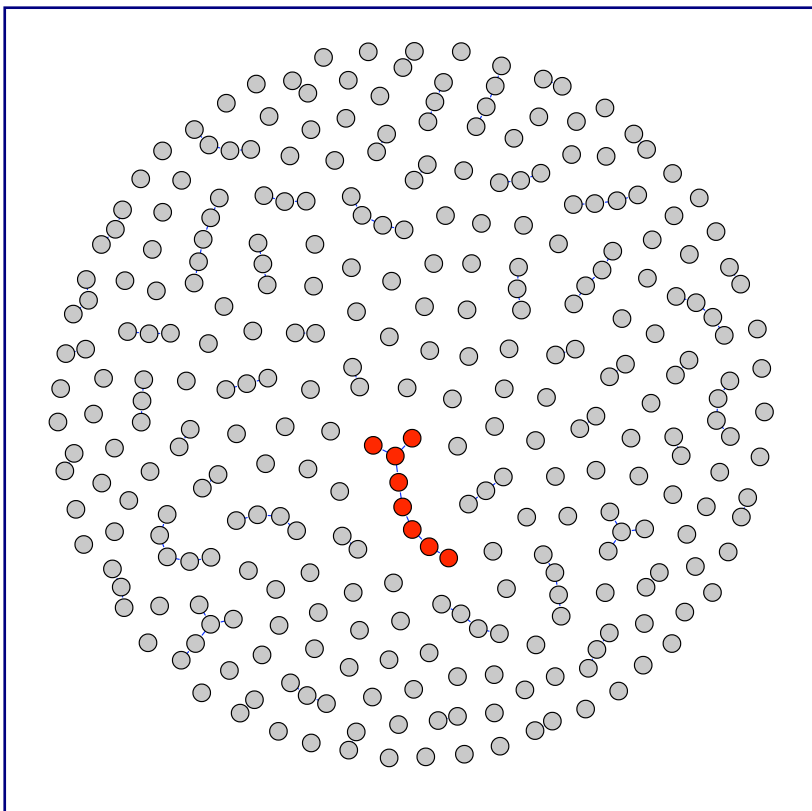
What does the resulting graph look like?
(Typical member of the ensemble)

Explicitly building $G(n, p)$

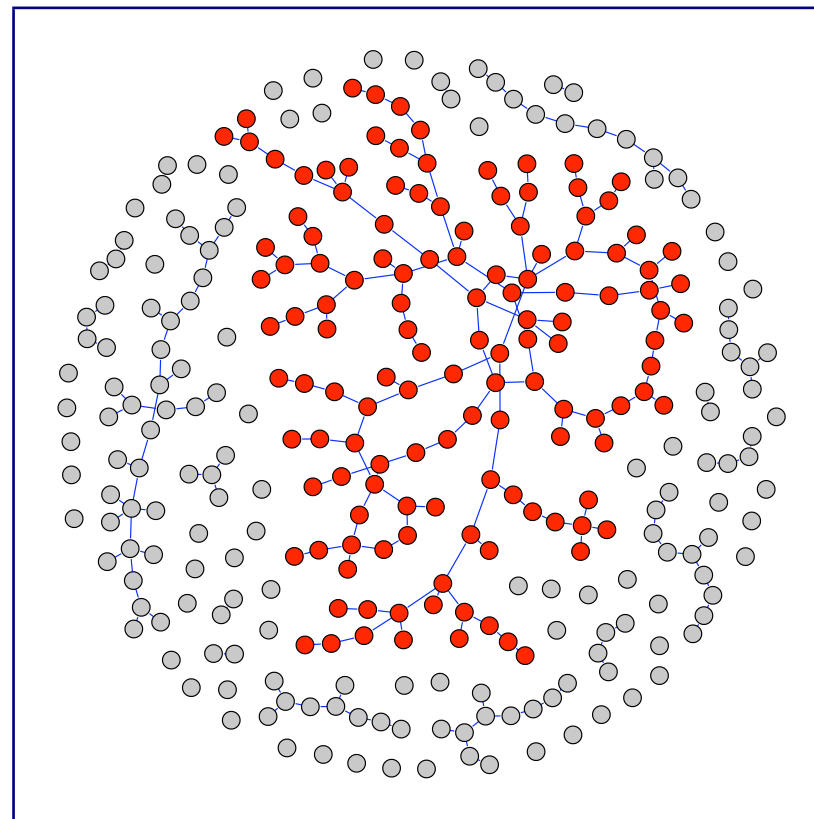
- Build a realization of $G(n, p)$ by the following graph process:
- Start with n isolated vertices.
- At each discrete time step, add one edge chosen at random from edges not yet present on the graph.
- At “time” t (i.e., at the addition of t edges), we have built a realization of $G(n, p)$ where $p = t/E$.
- This is a Markov process (build graph at time $t + 1$ from graph at time t).

Ben-Naim, **Krapivsky**, “Kinetic theory of random graphs”, *PRE*, 2005.

N=300



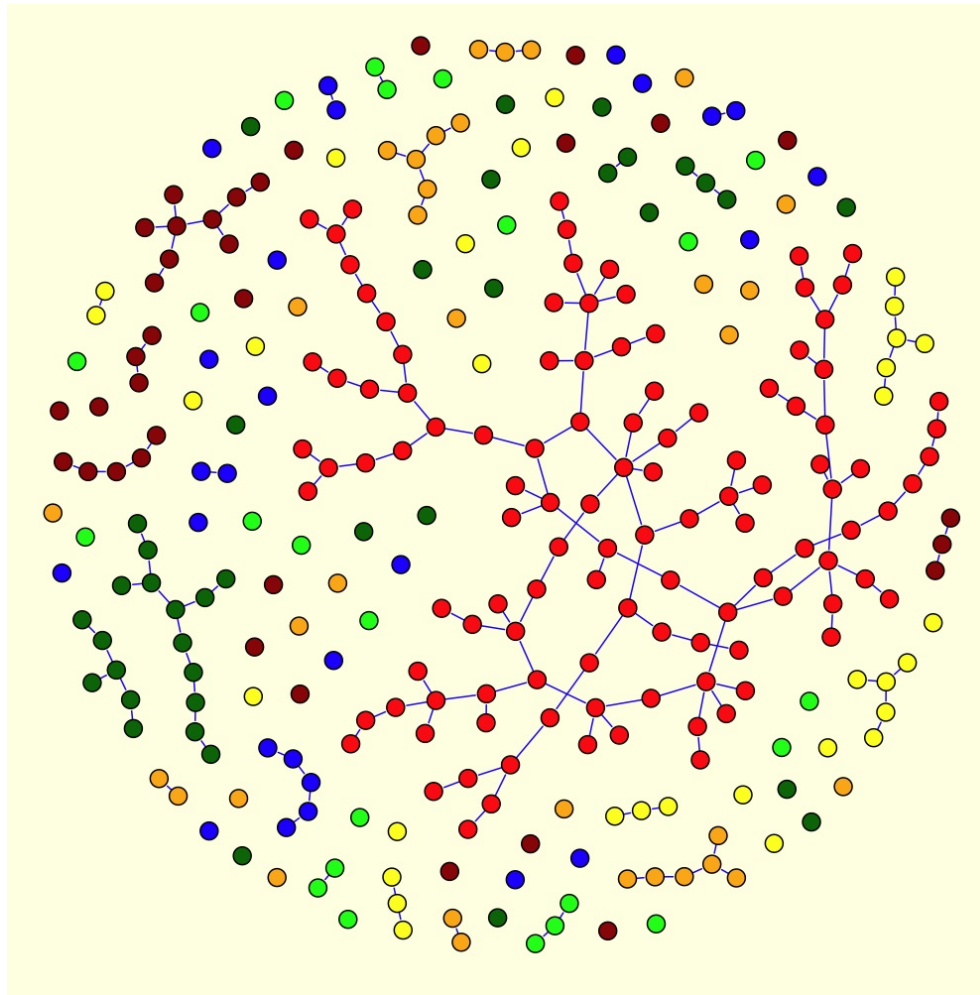
$$p = 1/400 = 0.0025$$



$$p = 1/200 = 0.005$$

Component

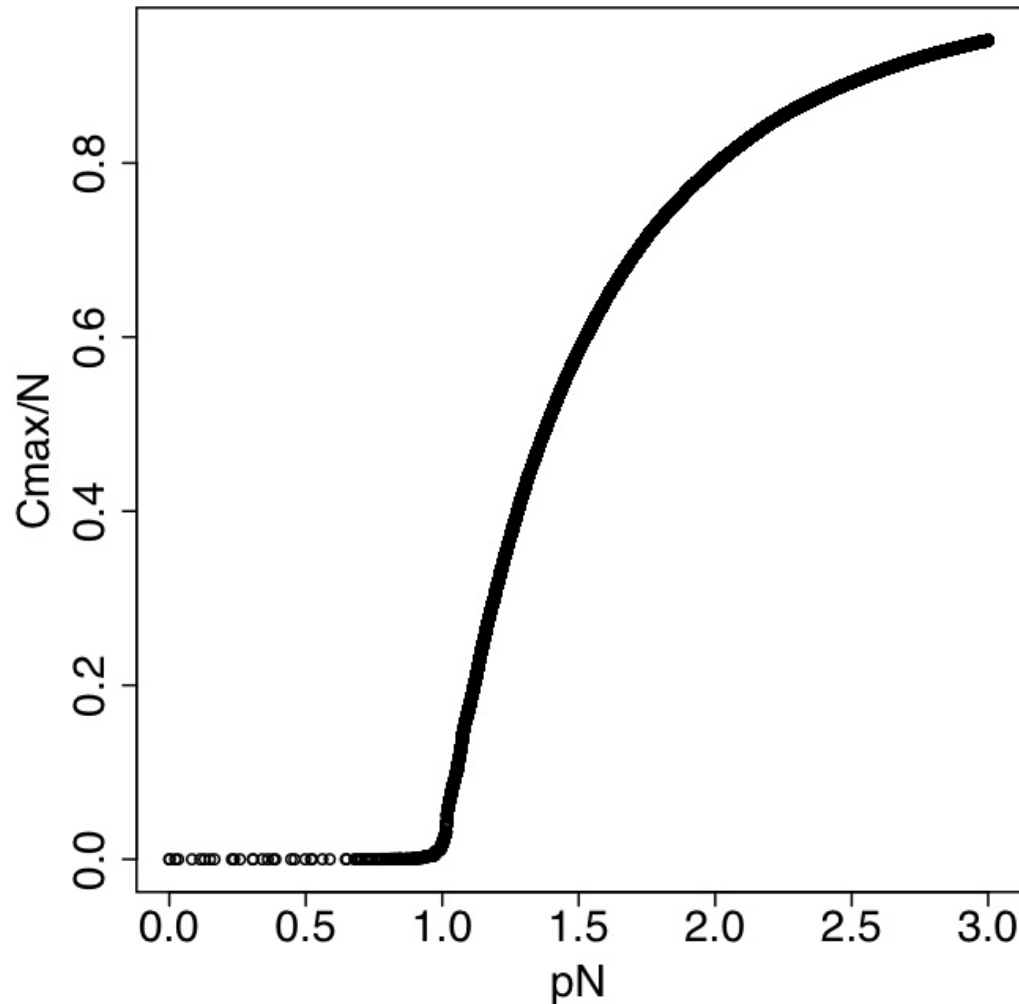
A **component** is a subset of vertices in the graph each of which is reachable from the other by some path through the network.



Behavior for small p

- Consider a realization $G(n, p)$ for $0 < p < 1$ and $n \rightarrow \infty$.
(A number of interesting properties of random graphs can be proven in this limit.
The $n \rightarrow \infty$ limit is also called the “thermodynamic limit”.)
- Let $C_{max}(p)$ denote the size of the largest component of $G(n, p)$ as a function of p .
- For small p , few edges on the graph. Almost all vertices disconnected. The components are small, with size $O(\log n)$, independent of p .
- Keep increasing p (or equivalently t in our model).
At $p = 1/n$ (i.e. $t = E/n$), something surprising happens:

Emergence of a “giant component”



- $p_c = 1/N$.
- $p < p_c$, $C_{\max} \sim \log(N)$
- $p > p_c$, $C_{\max} \sim A \cdot N$

(Ave node degree $t = pN$
so $t_c = 1$.)

Branching process (Galton-Watson); “tree”-like at $t_c = 1$.

A Phase Transition!

An abrupt sudden change in one or more physical properties, resulting from a small change in a external control parameter.

Examples from physical systems:

- Magnetization
- Superconductivity
- Liquid/Gas
- Bose-Einstein condensation

Giant component observed in real-world networks

- Formation reminiscent of many real-world networks.
“Gain critical mass”.
- Lower bound on emergence of epidemic outbreak.
- The giant component/Strongly Connected Component used extensively to categorize networks.

Phase transition in connectivity

- Below $p = 1/n$, only small disconnected components.
- Above $p = 1/n$, one large component, which quickly gains more mass. All other components remain sub-linear.
- Note the average node degree, z :

$$\begin{aligned} z &= (2 \times \#edges) / \#vertices \\ &= (2pn(n-1)/2) / n = pn(n-1)/n = (n-1)p \approx np. \end{aligned}$$

(Factor of 2 since each edge contributes degree to two vertices – each end of the edge contributes.)

Recall, expected number of edges, is $pn(n-1)/2$.

- At the phase transition, $z = np = 1$. The phase transition occurs when the average vertex degree is one!

Erdős-Rényi, a continuous, second order transition: Mean-field scaling behaviors

- Divergence of susceptibility: $\chi = \frac{\partial m}{\partial h} \sim |T - T_c|^{-\gamma}$
- Random graph “susceptibility” (second moment of the component sizes): $\chi = \sum_{i=1}^{\infty} i^2 n_i$
- For Erdős-Rényi, $\chi \sim |t_c - t|^{-\gamma}$, with $\gamma = 1$.

Power law correlation lengths and response functions →

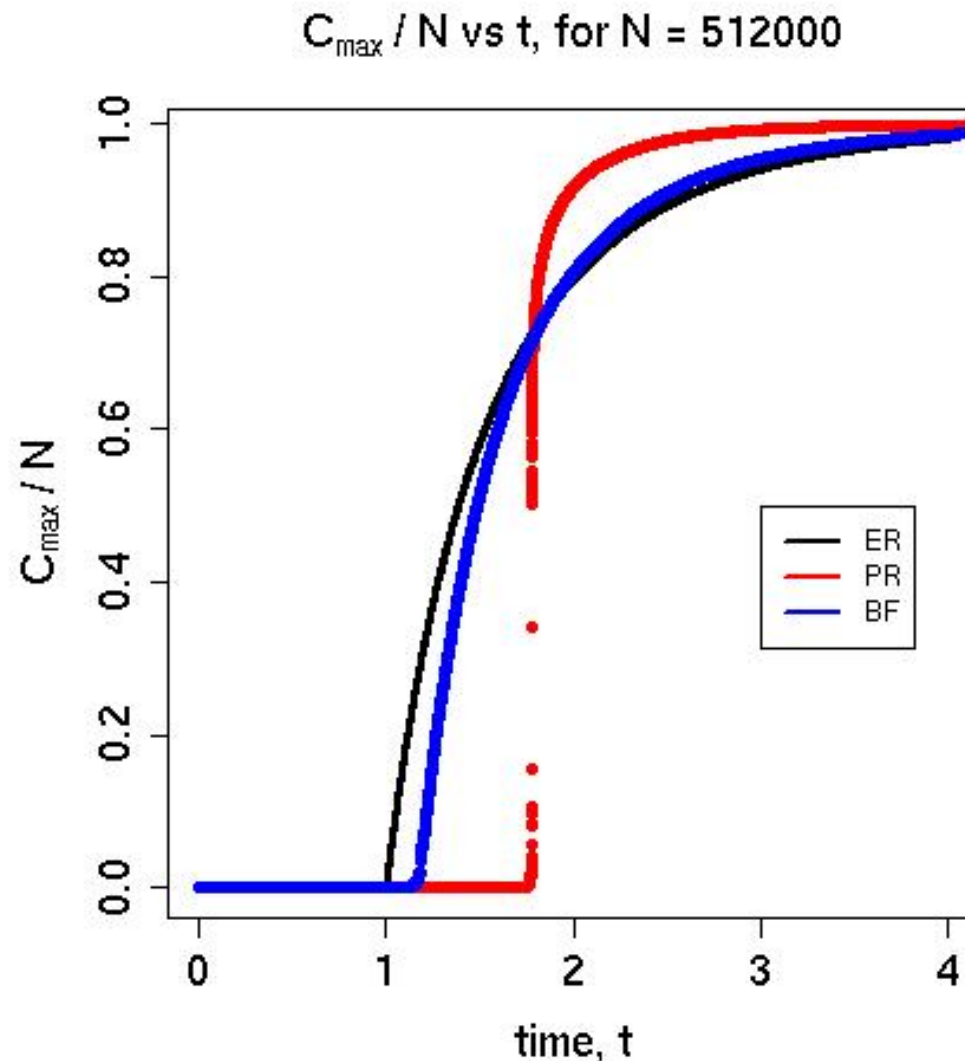
Potential EARLY WARNING SIGNALS

(e.g., Scheffer et al. *Nature* 461, 2009)

Is connectivity a good thing?

- Communication, transportation networks
- Spreading of a virus (human or computer)

Algorithms for suppressing the emergence of the Giant Component

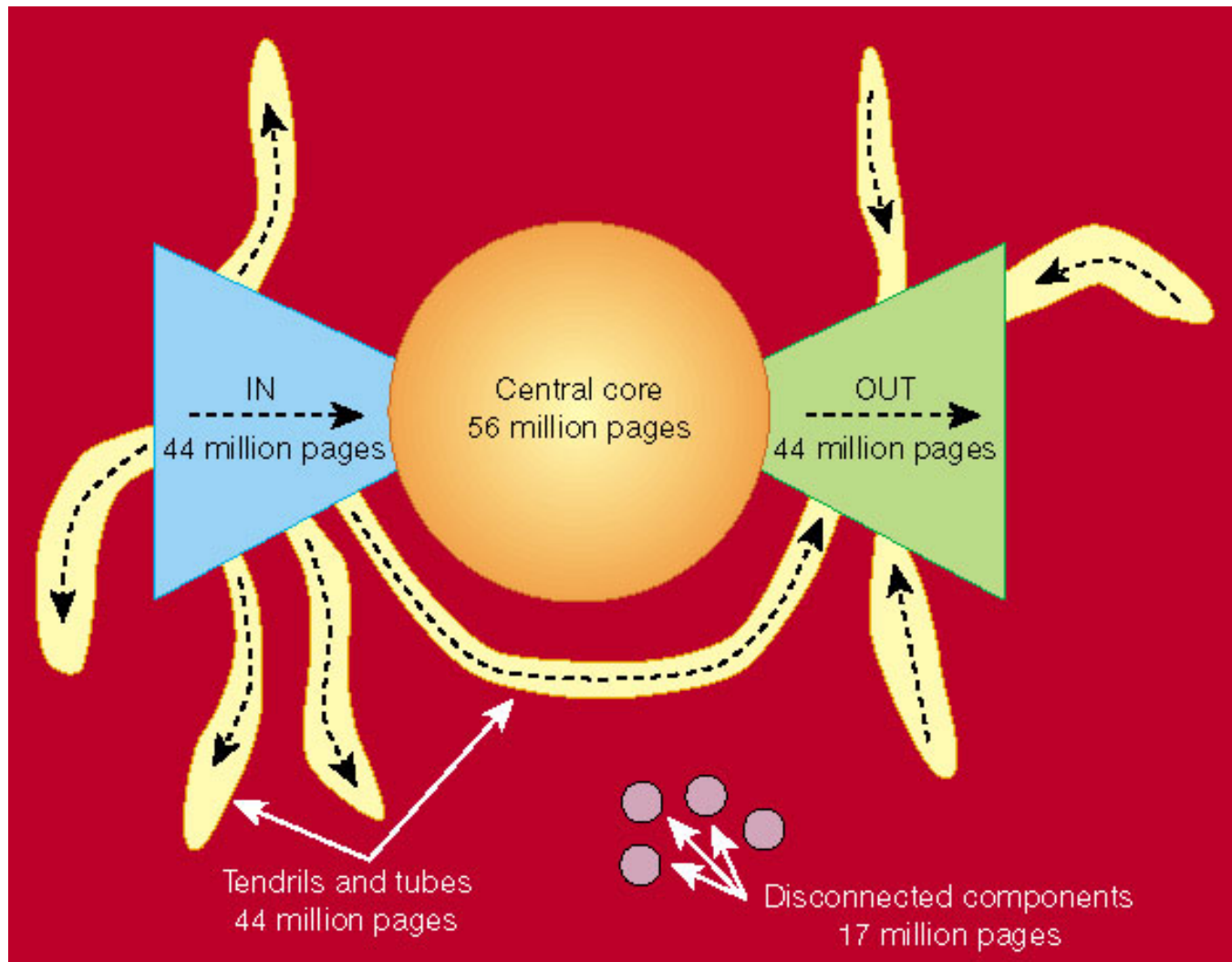


e.g. “Explosive percolation”, Achlioptas, D’Souza, Spencer,
Science, 2009.

Random graphs as real-world networks?

- What about degree distribution, clustering, assortativity....?
 - Shown later, Erdos-Renyi yields a Poisson degree distribution, but “configuration” models work around this.
 - Still need null models to match other properties.
- e.g., “Network Analysis in the Social Sciences”, S. P. Borgatti, A. Mehra, D. J. Brass, G. Labianca, *Science* **323**, 892-895, 2009.
 - Why would a real network look like a random one?
 - Local properties of nodes and edges, not statistics of the network.
- Developing the correct null models?

The giant component/Strongly Connected Component of the WWW



From "The web is a bow tie" Nature **405**, 113 (11 May 2000)

Summary: Terms introduced today

- Component
- Phase transition
- Degree distribution
- Graph diameter

Further reading on random graphs

- M. E. J. Newman review, pages 20-25. (Heuristic arguments)
- R. Durrett book, Chaps 1 and 2. (Technical proofs)
- B. Bollobás, *Random Graphs*, 2nd Edition, Cambridge U Press, 2001 (the seminal text on the mathematics of random graphs).

Project pitch – HW1a

- One page describing your idea. Submitted via Canvas and shared with the class.
- Skill sets to merge:
Domain specific questions / Methods / Data sets

Possible topic areas

- Transportation networks and flows; multi-modal transportation
- Open source software – e.g., social and technological networks in github
- Machine learning – e.g., bring network connectivity into binary classifiers
- Power grid modeling
- Opinion dynamics / social unrest / multiplex opinion dynamics
- Ranking in networks; especially temporal, multilayered, higher-order
- Multilayered and temporal macaque monkey networks
- Shocks and tipping points
- Extend standard metrics to multilayered, temporal, or higher-order networks
- Co-author and citation networks
- Food networks
- Recommendation systems
- Biological networks
- Terrorist networks
- See also class homepage “Projects” tab