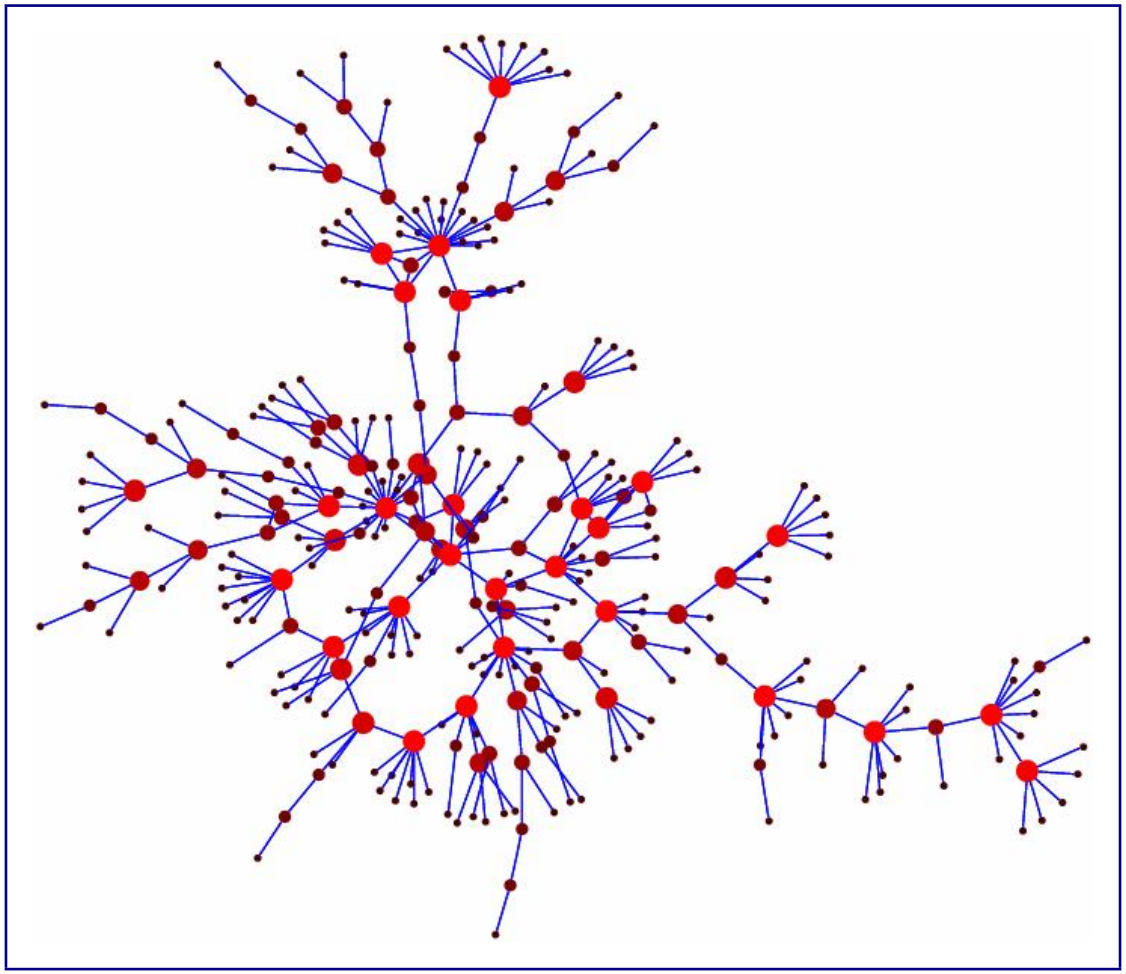
MAE 298, Lecture 10

May 4, 2006



“Percolation and Epidemiology on Networks”

Processes on networks

- Search for information
- Spreading processes

Interplay of topology and function

Epidemiology

- Understanding how diseases spread on networks
- Human diseases
- Computer viruses (typically spread via email networks)
  - Typically attached to an executable program.
  - Typically corrupt files on host computer
- Computer worms(spread directly from computer to computer via network connections)
  - Worms are self-contained.
  - Generally harm the network and consume bandwidth.

A general challenge is to understand how information fields flow on networks, even on dynamic networks!

- Multiple types of info flowing simultaneously
- Multiple length and time scales

Consider the spread of the avian flu

Occurs on a dynamic network with multiple length and time scales:

- Long length exchanges of the virus strains:
  – between migrating flocks,
  – between people flying on airplanes.
- Short length (local) exchanges of the virus strains.
- Exchange of additional information each interaction (health warnings, weather patterns, etc.) that influence future connectivity of network.

In addition to information flow on network,

need to consider effects of

- self-organization
- phase transitions (i.e., epidemic threshold)

Complications

- Multiple fields of information flowing simultaneously, each with its own length and time scale.
- Network is dynamic across multiple length and time scales.
- The information fields can interact and also influence the future network structure.
- Multiple networks involved, some layered on one another (email networks, airplane networks, fuel distribution networks, etc).
- Phase transitions.
- Self-organization.

Starting simply

Understand flow of one information field (i.e., a virus) on a static network.

(Even this is complicated)

- SIR (Susceptible, Infected, Removed)
- SIS (Susceptible, Infected, Susceptible)

- S = don't have the disease but can catch it if exposed.
- I = have the disease and can pass it on.
- R = recovered with permanent immunity (or "removed").

Traditional mathematical epidemiology

- β – probability of an S catching disease from an I.
- γ – probability of an I recovering and becoming an R.

Neglect any spatial structure, and assume *fully mixed* (i.e., any individual is equally likely to come into contact with any other).

- In graph theory terms, this would be the complete graph.
- Also called “mean-field” in physics.

The resulting rate equations:
The Kermack-McKendrick model:

[Kermack and McKendrick, "A Contribution to the Mathematical Theory of Epidemics." Proc. Roy. Soc. Lond. A 115, 1927]
[Anderson and May, "Population Biology of Infectious Diseases: Part I." Nature 280, 1979]

Three coupled ordinary differential equations:

1. $\frac{dS}{dt} = -\beta I S,$

2. $\frac{dI}{dt} = \beta I S - \gamma I,$

3. $\frac{dR}{dt} = \gamma I.$

Epidemiological threshold

$$\boxed{T_c = \frac{\beta S_0}{\gamma}}$$

- Where S_0 is initial size of susceptible population.
- For $T_c < 1$ disease dies out, $dI/dt < 0$.
  (An I infects less than one S before recovering or dying).
- For $T_c > 1$ disease will spread until full population gets infected, $dI/dt > 0$.
  (An I infects more than one S).

Incorporating network structure: Bond Percolation (Contact processes)

[Grassberger, "On the critical behavior of the general epidemic process and dynamical percolation", Math. Biosci., 63, 1983.]

- Assume randomly chosen initial carrier.
- Probability disease is transmitted corresponds roughly to the edge occupancy probability. Remember the Erdös-Renyi random graph, but here we are given an underlying network and are "activating" selected edges.

See Jeff Achter's homepage for java simulations of percolation on a lattice: http://www.math.colostate.edu/∼achter/

Bond percolation, cont.

- Look at distribution of cluster sizes. These correspond to extent of disease spread. Note all we get are the final S and R values. Says nothing about the dynamics! Just the final state.

- The percolation transition corresponds to the epidemic threshold. The size of the giant component corresponds to the size of the epidemic.

- How do we choose the underlying graph?

  – Almost every social network studied shows power law degree distributions.

  – The Internet has a highly right-skewed degree distribution.

  – Alternately, the small-world of Watts and Strogatz gives a different starting point.

  – Power law random graphs easier to analyze.

Behavior on random graphs:

Epidemic Threshold

- Infinite graphs

$\lambda \leq \lambda_1$	$\lambda_1 \leq \lambda \leq \lambda_2$	$\lambda_2 \leq \lambda$
extinction	weak survival	strong survival
	λ_1	λ_2

- Finite graphs

logarithmic survival time	polynomial survival time	exponential (super poly) survival time
	λ_1	λ_2

Power law random graphs: Configuration model

- Decide what degree distribution is desired.
- Generate a set of N isolated vertices with "stubs", matching the desired degree distribution.
- Connect stubs from randomly chosen vertices until all stubs have been matched up into edges.
- Can get exactly the power law desired.
  (Not possible with E-R random graphs, which have Poisson degree distributions, or Preferential Attachment graphs, which have power law, $p_k \sim k^{-\gamma}$, with $\gamma = 3$.)

Recall power laws

$$\boxed{p_k \sim k^{-\gamma}}$$

- For $\gamma > 1$ properly defined probability distribution.
- For $\gamma < 3$ the variance (thus also standard deviation) diverge to infinity. Can have finite mean (if $\gamma > 2$, but infinite variance).

This infinite variance plays a critical role.

Percolation/epidemic threshold on power law random graphs

[Callaway, Newman, Strogatz and Watts, "Network robustness and fragility: Percolation on random graphs", Phys. Rev. Lett., 85 (2000)]

- Use generating functions.
- Show that for power law networks with $\gamma < 3$ the percolation threshold, $T_c = 0$! So any one sick individual will infect a significant fraction of population (giant component exists even for $T_c = 0$.
- This results from the fact that disease multiplies exponentially if the variance diverges.
- Implications for disease spread?

Implications for disease spread?

- Are human contact networks really like power law random graphs?

- Yes, they have the power law degree disribution.

- But usually, also more structure:
  – Geographic correlation.
  – Degree-degree correlations.
  – High transitivity.

- Each of the three factors alone can make $T_c > 0$.

Developing a model that accurately captures human connectivity still in the works.

Immunization:
Coupling percolation and network resilience

- View vaccination as removing a particular set of vertices from the network.
- As we saw previously, removing the high-degree nodes from a power law random graph, quickly destroys connectivity.
- How to find these “hubs” in a social network, for instance a network for sexually transmitted diseases?

Identifying Hubs

- Want to sample *edges* rather than *nodes*.
- Choose node at random, probability of choosing node of degree k is $1/N$ (independent of k).
- Choose an edge at random, probability of it leading to a node of degree k proportional to kp_k.
- How to chose an edge at random?

Acquaintance vaccination

- Choose a person at random.
- Then choose a *friend* of that person to vaccinate.

[Cohen, ben-Avraham, and Havlin, "Efficient Immunization of Populations and Computers", Preprint 0207387 (2002); available from http://arxiv.org/abs/cond-mat/]

Show by computer simulation and analytic calculations that this is much more effective than random vaccination.

This type of acquaintance vaccination actually used to control small pox and foot-and-mouth ("ring vaccination")

How to model a real human population? (Using census data)

[Bansal, Pourbohloul, Meyers, “The Spread of Infectious Disease through Contact Networks”, Talk given at MSRI, April 2005.]

Not published, but a video can be viewed at:
http://angelina.msri.org/VMath/show_speakertalks?field_pid=900000033

Take actually census data from the city of Vancouver.

Constructing connectivity via census data

- Households
- Classrooms
- Businesses
- “Shopping”

Who to immunize?

Strategy one: Immunize the “hubs”

- Receptionists
- Bus drivers
- School teachers

This results in the least number of people becoming infected.

Strategy two: Immunize the most frail

- Elderly and children.

More people overall get infected, but less people overall die as a result of the disease!

Model of city of Vancouver
how to generalize?

- Random rewiring — see if results still hold.
- Compare with census data of other cities.
- Clearly this is a model for the local transmission of disease, but may be adequate.

Summary

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