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Abstract

Significant methodological progress has taken place to quantify the reliability of networked systems over the past decades. Both numerical and analytical methods have enjoyed improvements via a host of advanced Monte Carlo simulation strategies, state space partition methods, statistical learning, and Boolean functions among others. The latter approach exploits logic to approximate network reliability assessments efficiently while offering theoretical error guarantees. In parallel, physicists have made progress modeling complex systems via tensor networks (TNs), particularly quantum many-body systems. Inspired by the representation power of quantum TNs, this paper offers a new approach to efficiently bound network reliability (REL) classically. It does so by exactly solving a related network Boolean satisfiability counting problem (or #SAT_{NET}), represented as a TN, which upper-bounds general all-terminal reliability (ATR) problems by counting configurations in which all network nodes are connected to at least a neighbor. Our #SAT_{NET} counting outperforms state-of-the-art approximate counters for the same problem as shown for challenging two-dimensional lattice networks of increasing size. While the over-counting from #SAT_{NET} increases exponentially relative to the number of configurations that satisfy (ATR) or #REL_{AT}, the bias is predictable for ideal networks, such as lattices, and the upper-bound is guaranteed with 100% confidence---a desirable feature when other methods with error guarantees fail to scale. Clearly, our goal is not to solve the general stochastic network reliability problem, which remains a #P problem in the computational complexity hierarchy (i.e., a counting version of non-deterministic polynomial time [NP] problems for which there is no known polynomial time algorithms to find their solutions). Instead, we present a novel bounding technique for network reliability, which relies on exactly counting satisfiable configurations in the SAT_{NET} problem by using quantum computing principles. We offer a proof for the counting bound to hold in a connectivity ATR setting, and illustrate trends with cubic and lattice networks. Overall, the proposed method provides an alternative to available system reliability assessment approaches, and opens directions for future research, especially as discoveries in logic, algebraic projections, and quantum computation continue to accrue.

Keywords	Network reliability; Tensor networks; Satisfiability; Counting; Boolelan logic; Quantum computing.

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Department of Civil and Environmental Engineering
George R. Brown School of Engineering

April 25, 2017

Prof. Bruce R. Ellingwood
Editor-in-Chief
Colorado State University
Fort Collins, Colorado, USA

Dear Dr. Ellingwood,

Submission of Original Research Paper

On behalf of my co-authors, I would like to submit our research manuscript on "Quantum-Inspired Boolean States for Bounding Engineering Network Reliability Assessment" for peer review and possible publication in *Structural Safety*.

This study offers an alternative perspective to exactly upper-bound network reliability, which relies on representing networked systems as quantum tensor networks. Such algebraic representations, endowed with qubits and specialized Boolean operators enable us to cast network reliability as a counting problem, which offers advantages in memory space and computation time at least for some ideal network configurations.

The manuscript has neither been published nor submitted for publication elsewhere. We have followed the guidelines established by the journal in preparing the document, and all co-authors are aware and supportive of this submission.

Thank you for considering this manuscript as a possible contribution to *Structural Safety*.

Sincerely,

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Quantum-Inspired Boolean States for Bounding Engineering Network Reliability Assessment

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April 26, 2017

Abstract

Significant methodological progress has taken place to quantify the reliability of networked systems over the past decades. Both numerical and analytical methods have enjoyed improvements via a host of advanced Monte Carlo simulation strategies, state space partition methods, statistical learning, and Boolean functions among others. The latter approach exploits logic to approximate network reliability assessments efficiently while offering theoretical error guarantees. In parallel, physicists have made progress modeling complex systems via tensor networks (TNs), particularly quantum many-body systems. Inspired by the representation power of quantum TNs, this paper offers a new approach to efficiently bound network reliability (REL) classically. It does so by exactly solving a related network Boolean satisfiability counting problem (or $\#SAT_{NET}$), represented as a TN, which upper-bounds general all-terminal reliability (ATR) problems by counting configurations in which all network nodes are connected to at least a neighbor. Our $\#SAT_{NET}$ counting outperforms state-of-the-art approximate counters for the same problem as shown for challenging two-dimensional lattice networks of increasing size. While the over-counting from $\#SAT_{NET}$ increases exponentially relative to the number of configurations that satisfy (ATR) or $\#REL_{AT}$, the bias is predictable for ideal networks, such as lattices, and the upper-bound is guaranteed with 100% confidence—a desirable feature when other methods with error guarantees fail to scale. Clearly, our goal is not to solve the general stochastic network reliability problem, which remains a $\#P$ problem in the computational complexity hierarchy (i.e., a counting version of non-deterministic polynomial time $[NP]$ problems

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for which there is no known polynomial time algorithms to find their solutions). Instead, we present a novel bounding technique for network reliability, which relies on exactly counting satisfiable configurations in the SAT_{NET} problem by using quantum computing principles. We offer a proof for the counting bound to hold in a connectivity ATR setting, and illustrate trends with cubic and lattice networks. Overall, the proposed method provides an alternative to available system reliability assessment approaches, and opens directions for future research, especially as discoveries in logic, algebraic projections, and quantum computation continue to accrue.

1 Introduction

With the advent of community resilience principles in engineering (Bruneau et al., 2003; Barker et al., 2013; Bocchini et al., 2014; Cimellaro et al., 2016), it is critical to advance methods to quantify the performance of systems, especially when abstracted as networks as in the case of distributed infrastructure. Recent efforts have focused on the reliability side that contributes to resilience. These include smarter simulation and graphical methods, particularly those based on Markov-Chain Monte Carlo (MCMC) (Ching and Hsu, 2007; Zuev et al., 2015), Bayesian Networks (BN's) (Straub and Der Kiureghian, 2010; Gehl and D'Ayala, 2016), and statistical learning (Stern, 2015). Also, closed-form methods and their approximations continue to offer unparalleled insights to system reliability problems, although these often require customized treatment to make problems computationally tractable (Li and He, 2002; Song and Kang, 2009; Daly and Alexopolous, 2006; Dueñas-Osorio and Rojo, 2011; Kim and Kang, 2013).

The goal of the present study is to show an alternative approach for system reliability assessment that is rigorous and will have future practical appeal as quantum computation and the requisite science and engineering developments continue to consolidate. The new perspective relies on creating quantum tensor networks (TNs) that mirror general stochastic networked systems, but whose contraction (i.e., tensor products over all shared indices) yields a scalar with the true count of configurations that satisfy a related Boolean $\#SAT_{NET}$ problem—the latter problem, explained below, is a superset of all satisfiable all-terminal (AT) reliability configurations and counts configurations in which all nodes have at least one neighbor. As engineering networks typically do not contain hyperedges (links that join more than two end nodes) and have low degree (the number of links per node tend to be independent of network size), it is possible to perform the TN contractions efficiently and exactly via low-rank tensor products.

Recent research has linked quantum tensor networks to general $\#SAT$ (Garcia-Saez and Latorre, 2012; Biamonte et al., 2015), where computations are shown to remain generally $\#P$ if pursued with classical computers. If quantum computers are used instead, TN contractions yielding $\#SAT$ values for general networked systems (including hyperedges and non-constant node degree) can be approximated with additive error in polynomial time (Arad and Landau, 2010). Classically, $\#SAT$ and network reliability (REL) have been linked recently, offering multiplicative guaranteed approximations to the counting $\#REL$ assessment in polynomial time (Dueñas-Osorio et al., 2017). However, no research has established a connection between TNs, which encode quantum Boolean states, and network reliability directly, especially for exact bounding purposes. We establish such a bound for $\#REL_{AT}$ via quantum TNs and associated over-counting of satisfiable configurations for networked systems, $\#SAT_{NET}$, particularly for networks without hyperedges and low degree when links are stochastic. Hence, our proposed approach provides new algorithmic alternatives to engineering reliability analysts and decision makers. Each of the TN and SAT foundational concepts is briefly introduced in the subsections that follow.

1.1 Algebraic Tensor Networks (TN)

At their simplest, tensor networks are linked multidimensional arrays, where the rank of the tensors corresponds to the number of indices in them, such as a rank-2 tensor, which corresponds to the familiar matrix $A_{\alpha\beta}$ in two dimensions α and β (Landsberg, 2011). Linkages among tensors are important as they enable contractions over shared indices. For instance, the traditional matrix product $C_{\alpha\gamma}$ amounts to a contraction of index β as $C_{\alpha\gamma} = \sum_{\beta=1}^{d} A_{\alpha\beta}B_{\beta\gamma}$, where d represents the different number of values taken by the entries along the dimension indexed by β. In the reliability problems considered in this paper, $d = 2$ so as to capture binary states, although multi-state systems could be considered for $d \geq 2$. These tensorial objects have found successful applications in physics, particularly for representing quantum many-body system states and for computing efficiently on them (Yoran and Short, 2006; Markov and Shi, 2008; Orus, 2014). Although specific developments in physics are beyond the scope of the current discussion, some recent advances to study strongly correlated systems with non-local interactions in high dimensions, substantiate the capabilities and insights offered by tensor networks (Murg et al., 2010). In fact, their appeal resides in that the tensor network language is intuitive as it is a generalization of matrices, which are all familiar to scientists and engineers.

For TNs to offer an alternative perspective in network reliability assessments, this study requires that specialized quantum-inspired tensors be used within a network layout of interest, thus forming a quantum TN. However, this network should also be consistent with the logic-based SAT formulas that enable the counting that contains reliable network configurations. Hence, the shared indices across tensors must represent edges of the TN (which are entangled tensors), the gate tensors represent the nodes, and their TN graphical representation shows system connectivity. Note that tensor and connectivity constraints can be handled via tensor contractions (Figure 1). To establish what the tensors of the network should be to allow for contractions that encode information relevant to engineering reliability, the classical Boolean satisfiability problem SAT needs to be invoked. Interestingly, SAT on its own continues to show practical success despite its proven computational hardness (Malik and Zhang, 2009).

1.2 Boolean Satisfiability (SAT)

The SAT problem relates to the satisfiability of a Boolean formula via an assignment of variables, such that the formula evaluates to 1 or True, after applying the necessary logic operators AND, OR, and NOT in the formula, typically represented by $\wedge$, $\vee$, and $\neg$, respectively. A common way of representing Boolean formulas for SAT is via their conjunctive normal form (CNF), which highlights the conjunction of clauses, each with a given number of literals or variables (Crama and Hammer, 2011):

$$f = \bigwedge_{i=1}^{m} (x_{i1} \vee x_{i2} \cdots \vee x_{ik_i}), \tag{1}$$

where f is the CNF formula, m the number of clauses, and each clause with up to k_i variables, for a total of n distinct variables in the formula, where $n \leq \sum_i^m k_i$. Note that this CNF expression is evaluated mathematically via products of sums, which are related to the more canonical sums of products, or disjunctive normal form (DNF), used in system reliability to represent reliability polynomials (Provan and Ball, 1983). Also, CNF formulas can encode the structure of general networks or graphs (Samer and Szeider, 2010), so it is convenient to imagine one such formula graphically for a simple system composed of link elements in parallel connected in series (Figure

2), where clauses are nodes and variables are links (using the so-called incidence or bipartite graph representation). The associated $_{NET}$ formula of the graph is $f_1 = (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3 \vee x_4 \vee x_5 \vee x_6) \wedge (x_4 \vee x_5 \vee x_6) = (x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6)$ after removing automatically satisfied clauses. This formula with two clauses and six variables can take up to 2^6 variable assignments, of which 49 configurations are satisfiable, such as when all variables equal to one, and 15 configurations are non-satisfiable, such as when all variables equal to zero. These configurations contain familiar information, such as sets of paths and cuts in network reliability. In fact, the series-parallel topology of the network in Figure 2 guarantees that every SAT_{NET} solution to f_1 corresponds to a path that contributes to all-terminal connectivity reliability. For other network topologies, the CNF formula for REL_{AT} is as hard to obtain as solving the problem itself. Hence, we explore the CNF formula for SAT_{NET}, whose counting of satisfiable configurations $\#SAT_{NET}$ approximates $\#REL_{AT}$ by over-counting path sets that determine AT connectivity reliability.

Observation 1. *Any network with m nodes and without hyper-edges requires at least (m-1) edges to satisfy all-terminal connectivity reliability (ATR) at least with one configuration counted in $\#REL_{AT}$. The same network requires (m/2) edges to satisfy at least one CNF formula for the network satisfiability SAT_{NET} problem, where nodes are clauses and edges variables. As for connected networks with $(m \geq 2)$, we have that $(m-1) \geq (m/2)$, thus there are more satisfiable configurations for the counting $\#SAT_{NET}$ problem than for the counting $\#REL_{AT}$ problem as every configuration for REL_{AT} automatically satisfies the SAT_{NET} problem, but not the converse.*

1.3 Quantum Boolean States for Network Reliability

What remains is to connect the notions of REL_{AT} to quantum TNs (as the main contribution of this research), where SAT_{NET} formulas can be evaluated via TNs composed of nodes and links that are quantum Boolean states to be contracted, where such TN contractions yield $\#SAT_{NET}$ counts to approximate $\#REL_{AT}$. Note that contractions can be performed approximately in quantum computers in polynomial time for any TN configuration, as well as exactly on classical computers for restricted network topologies—node and link layouts without hyper-edges and low tree-width that are fortunately common to engineering networks. Also, throughout this paper, links are stochastic failing with probability p_{f_e} for $e \in E$ (where E is the set of edges in the network), while nodes remain reliable, but a generalization is possible by adding new edges per node to treat them stochastic as well.

The reminder of the manuscript is as follows: Section 2 discusses quantum Boolean states and CNFs as tensor networks. Section 3 offers the proposed process to evaluate TNs of quantum Boolean states, which are consistent with SAT_{NET} formulas, and themselves are set to count configurations that satisfy ATR problems ($\#REL_{AT}$). Section 4 shows how simple network topologies enable the study of logic-based quantum tensorial network reliability. Section 5 presents example applications, analysis of results, and paths for extensions, along with an illustration of the bound for network reliability using cubic and lattice systems. Then, conclusions revisit the connection between network reliability and quantum TNs, as enabled by SAT_{NET} formulas constrained by network layouts, to then provide ideas for additional research on quantum-based engineering system reliability.

2 Logic-Based Tensor Network Representations

To explicitly link SAT_{NET} formulas to tensor networks (TNs), and subsequently to REL_{AT} expressions, quantum states are essential as they capture classical system states, along with their superpositions. For instance, a single quantum state of a quantum bit, or qubit, could be in state 1 or state 0 (as classical bits are), or in both at the same time (this is a key difference with respect to classical configurations). We adopt the bra-ket Dirac notation here, which allows for a compact representation of quantum states. For instance, a single qubit in state 1 is represented as a ket $|1\rangle$, while two qubits in states 1 and 0, respectively, are denoted by $|10\rangle$. Their bras are represented as $\langle 1|$ and $\langle 10|$, respectively, which equal to the complex conjugate transposes of their kets. Although quantum states are vectors in complex Hilbert spaces, they can be understood in this paper as vectors encoding the state of a system (e.g., the binary state of elements in a network). Thus, they can be expressed as a sum of orthogonal basis vectors, the same way a vector in a Cartesian space can be decomposed into its unit vectors (Susskind and Friedman, 2014). For instance, a general superposition state of a single qubit A can be denoted in its basis as $|\Psi_A\rangle = \alpha |0\rangle + \beta |1\rangle$, where the coefficients α and β correspond to the wave function amplitudes of states 0 and 1, which are complex numbers, but whose magnitudes capture the probability of the system to be observed in states 0 or 1, respectively, if measured (note that $|\alpha|^2 + |\beta|^2 = 1.0$). For the case of two qubits,

A and B, their general state in Dirac notation is $|\Psi_{AB}\rangle = \alpha|00\rangle + \beta|10\rangle + \gamma|01\rangle + \delta|11\rangle$, where $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1.0$. A key to linking SAT_{NET} and TN concepts is that quantum states, such as $|\Psi_{AB}\rangle$ or more general ones, can all be represented as tensors.

The provided superposition notation enable us to write the state of a general system, such as a network, as the sum of amplitudes times their basis states. Hence, a single bitstring $\mathbf{x}$ of length n, with an assignment of variables in $\{0, 1\}$, represents a particular state of the n superposed qubits, containing the 2^n configurations that satisfy and do not satisfy a given CNF formula. However, as the collection of all bitstrings are orthogonal, their unnormalized amplitudes relate to their contribution to the logical evaluation of the CNF SAT_{NET} formula $f(\mathbf{x})$, in the form of Equation (1), mapping the space of $\mathbf{x} \in \{0, 1\}^n$ onto the space of $f(\mathbf{x}) \in \{0, 1\}$. Hence, the quantum state $|\Psi_f(\mathbf{x})\rangle = \sum_{\mathbf{x}} f(\mathbf{x})|\mathbf{x}\rangle$ is a superposition of states whose non-zero amplitude terms represent the satisfying variable assignments of formula f, and thus referred to as a Boolean quantum state. In fact, every quantum state written in a local basis with amplitude coefficients in $\{0, 1\}$ gives rise to a Boolean relation (Morton and Biamonte, 2012). The satisfying assignments, in the context of a network such as the one in Figure 2 or any other general layout, also contain all possible configurations that contribute to connectivity-based ATR.

To exemplify the notions covered so far, take a simpler version of Figure 2, where only variables (edges) x_1, x_2, x_4, and x_5 are involved (Figure 3). In the CNF context, there are four variables, two non-redundant clauses, and $2^4 = 16$ configurations of variables. The CNF formula of this graphical representation involves clauses C_1 and C_3 and corresponds to $f_S = (x_1 \vee x_2) \wedge (x_4 \vee x_5)$. Here, $\#SAT_{NET} = 9$, indicating the number of variable assignment sets that satisfy the formula—in this case the same variable assignment sets contribute to $\#REL_{AT} = 9$. In terms of a Boolean quantum state, f_S can be written as a superposition of four qubits (one per variable): $|\Psi_{f_S}\rangle = 0|0000\rangle + 0|1000\rangle + 0|0100\rangle + \cdots + 1|1011\rangle + 1|0111\rangle + 1|1111\rangle$. Once again, each of the basis ket vectors with non-zero amplitude corresponds to a configuration of variables that contributes to $\#REL_{AT}$. The key insight is that a quantum computer with n qubits could in principle hold all 2^n configurations as a Boolean superposed state and count relevant configurations upon measurement in quantum polynomial time (exactly for small CNF formulas, and approximately within additive error for any network size or topology and associated CNF formula) (Arad and Landau, 2010). Measuring information from such superposed states remains a science and engineering challenge, but progress is steady towards the quantum computer goal even with existing semiconductor technologies (Van Meter and Horsman, 2013; Hornibrook et al., 2015). In this paper, we exploit the fact that TNs allow to classically simulate certain quantum systems, particularly their SAT properties when describable by networks whose links are not hyperedges (as is the case for most engineering network layouts where the clause-to-variable ratio, or node-to-link ratio, tends to be smaller than 1.0 given that links connect to two nodes).

3 Counting by Tensor Network Contractions

The previous background prepares us to use the language of quantum TNs to handle CNF formulas for networked systems (SAT_{NET}). The key is to determine the form of the tensors to represent CNF formulas and operate on them as they encode solutions to $\#SAT_{NET}$ and bound $\#REL_{AT}$. While it is known that efficient computations are possible in classical computers for special networks with tree-like structure, or for quantum networks with exponentially small numbers of COPY tensors (defined subsequently) (Fischer et al., 2008; Biamonte et al., 2015), we later provide a way to evaluate $\#SAT_{NET}$ exactly and bound $\#REL_{AT}$ for networks without hyperedges.

To specifically see the role of TNs in our reliability assessment goal, we build upon algorithms for general $\#SAT$ (Biamonte et al., 2015). The goal is to construct a tensor network from the network topology at hand, such that its nodes and links are tensors consistent with a SAT formula $f(\mathbf{x})$ that captures the ATR problem of interest. The process is as follows (further illustrated with an example and figure after the next five steps):

Quantum-inspired counting bound of reliable network configurations

(1) Assign a binary tensor $\langle 0| + \langle 1|$ to each variable x_i of the CNF formula $f(\mathbf{x})$ that captures SAT_{NET} for a network layout of interest. Variables take values in $\{0, 1\}$, and represent links in the network (note that nodes could also be included as binary variables by enlarging the network with necessary nodes and links to make every original node a link).

(2) Create k copies of variable x_i if it appears in k clauses, excluding clauses automatically satisfiable. Perform this step by making use of a special COPY tensor, which is the outer product of the possible states of variable x_i and their kth tensor power as $|0\rangle \langle 0|^{\otimes k} + |1\rangle \langle 1|^{\otimes k}$. Note that the

outer product essentially projects the ket into the tensor power of the bra, thus creating a higher dimensional operator.

(3) Connect each variable x_i to its COPY tensor, so that there are enough qubits to feed into each of the SAT_{NET} clauses (network nodes) in which each variable appears (associated incident edges). This is done by generating new tensors via tensor products of variables and COPY tensors consistent with the network topology.

(4) Perform the outer tensor product of the tensors in every clause (which are tensors that function as logic gates: evaluating true when their qubit assignments satisfy the clause and false when they do not), with the binary tensors $\langle 0|$ or $\langle 1|$ corresponding to the true or false states of the clauses (which are OR gates). Then, perform the tensor product of the COPY-based tensors from step (3) with the clause-based tensors just obtained. While the process thus far is algorithmic, note that quantum logic gates could be physically realizable via quantum circuits, and likely implementable in actual hardware of the future (Johnson et al., 2013; Valiron et al., 2015). In this study we remain algorithmic; thus, this step essentially contracts links to nodes, and aggregates nodes of the original network based on SAT_{NET} constraints—essentially a tensor that now possess a count of satisfiable and unsatisfiable configurations based on all clause and variable contributions.

(5) Link the current tensor network contraction to a tensor $|1\rangle$ via another tensor product, such that the overall tensor contraction evaluates to a post-selected desired scalar outcome of overall satisfiability (i.e., summing over all the SAT_{NET} configurations that matter as they satisfy an overall AND gate which matches or upper-bounds $\#REL_{AT}$ depending on the topology of the network of interest).

Figure 4 showcases these previous five steps for the original network in Figure 3, where every object is a tensor, and their connections flag contractions via tensor products.

To illustrate the previous steps further, let us discuss the actual contraction process in Figure 4 corresponding to the TN for Figure 3. Here, each of the variables to the left is a tensor, the COPY tensor is the dark dot, and each clause is a tensor operator, or implementable OR gate (Valiron et al., 2015). In particular, the contractions in *Steps 1-3* run as follows: Each of the left-hand side tensor variables represents a qubit (or an edge of our network), and thus each variable is on a superposition ($\langle 0|+\langle 1|$), which undergoes a tensor product with its COPY tensor ($|0\rangle\langle 0|^{\otimes 1}+|1\rangle\langle 1|^{\otimes 1}$) to reproduce the link-to-node TN connectivity encoded in the SAT_{NET} formula f_S. Recall that each state in the Dirac notation can be written as a vector; hence, for further clarity in relation to classical linear algebra, the previous operation is set up using the algebraic convention for binary bits, where the column kets are $|0\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ and $|1\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$, while their row bra counterparts are the transposed complex conjugates of the kets $\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $\langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}$. The partial tensor network contraction under consideration is thus:

$$\begin{aligned}\big(\langle 0| + \langle 1|\big)\big(|0\rangle\langle 0|^{\otimes 1} + |1\rangle\langle 1|^{\otimes 1}\big) &= \Big(\begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \end{pmatrix}\Big)\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}\right] \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix}. \end{aligned} \tag{2}$$

Then, to complete *Step 3*, the previous result for the first variable x_1 and its COPY tensor undergoes a tensor product with the other variables in the same clause, in this case x_2, such that $\big(\langle 0| + \langle 1|\big)\big(|0\rangle\langle 0|^{\otimes 1} + |1\rangle\langle 1|^{\otimes 1}\big) \otimes \big(\langle 0| + \langle 1|\big)\big(|0\rangle\langle 0|^{\otimes 1} + |1\rangle\langle 1|^{\otimes 1}\big) = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$. Afterward, *Step 4* contracts the previous result with the clause tensor, which takes the form of a Boolean gate. This tensor is equivalent to the following operator: $\sum_{\mathbf{x}\in\{0,1\}^{k_i}} |\mathbf{x}\rangle\langle \phi_i(\mathbf{x})|$, where $\phi_i(\mathbf{x})$ is a binary tensor that evaluates the truth of the ith clause for a given bitstring $\mathbf{x} \in \{0,1\}^{k_i}$. Truth per clause in our context corresponds to the satisfaction of a clause (which is satisfied via an OR evaluation of its variables). This latter tensor in matrix notation for the first clause is as follows (note the advantage of the compact Dirac notation relative to the matrix notation):

$$\begin{aligned} C_1 &= |00\rangle\langle 0| + |10\rangle\langle 1| + |01\rangle\langle 1| + |11\rangle\langle 1| \\ &= \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} \\ &= \begin{pmatrix}1 & 0\\0 & 1\\0 & 1\\0 & 1\end{pmatrix}. \end{aligned} \tag{3}$$

Then, *Step 4* further contracts the variables and their COPY tensors $\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}$ along with their clauses in Equation 3. This leaves a 1×2 tensor for the case under consideration equal to $\begin{pmatrix}1 & 3\end{pmatrix}$. Note that this result is for all contractions related to C_1. But an identical contraction results for C_3 in this case, so that both are combined via a tensor product to complete *Step 4*, as $\begin{pmatrix}1 & 3\end{pmatrix} \otimes \begin{pmatrix}1 & 3\end{pmatrix} = \begin{pmatrix}1 & 3 & 3 & 9\end{pmatrix}$. Finally, as the interest is in evaluating the satisfiability of the overall TN, the current contracted results $\begin{pmatrix}1 & 3 & 3 & 9\end{pmatrix}$ are further contracted in *Step 5* with the desirable true value of each clause via its post-selection across clauses $|1\rangle^{\otimes 2}$, which in matrix notation is $\begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix}^T$. Thus, the overall tensor network contraction evaluates to $\begin{pmatrix}1 & 3 & 3 & 9\end{pmatrix} \otimes \begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix}^T = 9$, which is exactly the number of satisfiable (SAT_{NET}) configurations or configurations that upper-bound $\#REL_{AT}$ for the network in Figure 3—in this case $\#SAT_{NET}$ and $\#REL_{AT}$ are equal. Also, note that if the tensor product of $C_1 \otimes C_3$ is performed first (before the contractions of clauses with their variables and COPY tensors), then the resulting quantum Boolean gate explicitly contains the states of the entire network in Figure 3, including specific satisfiable configurations (the terms evaluating to truth that have an outer product with $\langle 1|$). For completeness, such an overall tensor on four qubits (variables) is $C_1 \otimes C_3 = |0000\rangle\langle 0| + |0001\rangle\langle 0| + \cdots + |0111\rangle\langle 1| + |1111\rangle\langle 1|$, where the 9 satisfiable configurations become self-evident as they are the ones projected onto $\langle 1|$.

The key message to engineering reliability analysts is that the use of quantum Boolean states of SAT_{NET} formulas provides an opportunity to exactly upper-bound the all-terminal connectivity reliability of networks in a new way, especially as quantum computers and associated algorithms continue to advance (Rieffel and Polak, 2014). Before quantum computations become mainstream, tensor networks still provide the machinery to classically simulate what a quantum system would do (Biamonte et al., 2011). We show in the next section that TNs can be contracted efficiently and exactly when evaluating $\#SAT_{NET}$ for networks without hyper-edges and low tree-width (independent of the size of the network)—which correspond to practical networks in engineering reliability. The disadvantage is that $\#SAT_{NET}$ only upper-bounds $\#REL_{AT}$.

4 Special Networked Systems

Regularity in the CNF formulas translates into structured topologies for networks and vice versa, which in turn can be systematically described by tensor networks that encode Boolean quantum states of a system. Taking for instance the CNF formula associated to a simple planar network, as in Figure 5, it is clear that each clause (or node) has degree three given the three incident edges per clause, and each edge represents a variable, which can be in more than one clause (two clauses in this maximally planar graph). The CNF formula for this regular network with four clauses (nodes) and six variables (links), where clauses have three literals, solves a special case of the so-called 3SAT problem. Here, the node-to-link ratio is < 1.0, and its SAT_{NET} formula corresponds to $f_P = (x_1 \vee x_2 \vee x_6) \wedge (x_1 \vee x_3 \vee x_5) \wedge (x_2 \vee x_3 \vee x_4) \wedge (x_4 \vee x_5 \vee x_6)$. The insight is that satisfiable variable configurations here also provide information for upper-bounding the number of configurations that contribute to the ATR problem.

Applying the five steps listed in the previous section to create the TN associated to the real network in Figure 5, the analyst can establish the Boolean superposition state of $2^6 = 64$ configurations for this system on six binary variables. Also, the contraction of the TN can count the number of satisfiable configurations, which for this network is 41 out of 64 when focusing on edges with binary states, thus upper-bounding the 38 satisfiable configurations of the associated ATR problem. In particular, *Steps 1-3* are equivalent to the tensor product of each of the variables (links) as qubits amplified by their COPY tensor. More general notation can also be introduced: for instance, for a single variable x_1, denote $Q^{i_{x_1}}_{\alpha_1,\alpha_2}$ the tensor that captures the possible variable binary states $\{0,1\}$ recorded in indicator i_{x_1}, as well as its degree 2 COPY tensor linking it to clauses 1 and 2: $((\langle 0| + \langle 1|)(|0\rangle \langle 0|^{\otimes 2} + |1\rangle \langle 1|^{\otimes 2})$, which in matrix notation corresponds to $\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$. Performing the tensor product across the six variable tensors $Q^{i_{x_1}}_{\alpha_1,\alpha_2} \otimes Q^{i_{x_2}}_{\beta_1,\beta_2} \cdots \otimes Q^{i_{x_6}}_{\zeta_1,\zeta_2}$, results in a $1{\times}4\,096$ tensor $\begin{pmatrix} 1 & 0 & 0 & 1 & \cdots 1 & 0 & 0 & 1 \end{pmatrix}$. Then, *Step 4* focuses on the gate tensors per clause, which in this problem when combined they correspond to the AND gate projector of the four clauses of the network. For instance, the first clause, C_1, is represented by the three qubit projector for variables x_1, x_2, and x_6 connected to the clause via ancillary indices α_1, β_1 and ζ_1. These indices (in Greek alphabet order for convenience here) are associated to each variable, while their respective subindices represent one of the two clauses each variable feeds into, thus capturing all their possibilities in a single Boolean operator. For a single clause, the operator is $|000\rangle \langle 0| + |100\rangle \langle 0| + \cdots + |111\rangle \langle 1|$. Then, the tensor product across projected clauses $C_1 \otimes C_2 \otimes C_3 \otimes C_4$ yields a $4\,096{\times}16$ tensor. Finally, the contraction of the tensor with all variables Q and the tensor with all projected clauses C via tensor products (where shared indices collapse in the sum of products), results in a quantum Boolean state tensor with all the relevant configurations at hand. Hence, when the latter tensor is contracted with the post-selection tensor required in *Step 5*, or $|1\rangle^{\otimes 4}$, it yields the number of satisfied assignments, which equals to 41 as expected. Note that the large tensors composed of Qs and Cs terms are constructed here by contracting variables only as well as clauses only first, so as to highlight the availability of a superposed quantum Boolean state with the satisfiable solutions explicitly revealed. However, contractions are more effective as amplified variables and projected clauses contract based on their connectivity, where variables and COPY tensors connected to projected clauses are contracted based on edges, so that the analysts only have to handle low-rank tensors.

Specifically for the problem at hand, rank-4 tensors with 16 states are the largest ones to handle when contracting variables and COPY tensors with clauses, compared to the rank-12 tensor with 4 096 states when contracting variables and COPY tensors separately from clauses. The larger tensor does reveal the specific satisfiable configurations, while the low-rank tensor approach is a trade-off that effectively counts satisfiable configurations but makes it harder to unravel individual configuration details. To facilitate the low-rank tensor contraction and counting further, it is possible to specify a priori the form for the individual variable and clause tensors, Q and C, so as to efficiently capture the desired Boolean superposition state that contains satisfiable solutions (Garcia-Saez and Latorre, 2012), and thus upper-bound the associated reliable configurations. Specifically, for planar networks as in Figure 5, which are equivalent to a 3SAT problem where every clause has three literals, and to upper-bound the ATR of this cubic graph, the following tensors apply:

$$Q^{i_{x_j}}_{\omega_1,\omega_2} = \begin{cases} 1, & \text{if } i_{x_j} = \omega_1 = \omega_2, j = 1, \ldots, n, \text{and } \omega = \alpha, \beta \ldots, n^{th} \text{ Greek symbol} \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

$$C_l^{\alpha,\beta,\gamma} = \begin{cases} 0, & \text{if } (\alpha = 0, \beta = 0, \gamma = 0) \in \omega, l = 1, \ldots, m \\ 1, & \text{otherwise} \end{cases} \tag{5}$$

These tensors have low ranks individually (two and three in this specific case), as shown by their ancillary Greek sub- or super-indices, so the contraction of the TN for the planar network in Figure 5 can be handled more efficiently. Specifically, this a priori specification of tensors for the network at hand requires handling $(n+m)$ low-rank tensor products whose maximum dimension do not exceed four. This saves space and also computation time. Implementing the tensor product, and pairing tensors of the TN such that common indices contract, yields the same result as before: 41 satisfiable configurations, without having to pre-specify any quantum Boolean superposition state coefficients. One possible economical contraction is as follows: $Q^{i_{x_1}}_{\alpha_1,\alpha_2} \otimes Q^{i_{x_2}}_{\beta_1,\beta_2} \otimes C_1^{\alpha_1,\beta_1,\zeta_1} \otimes C_2^{\alpha_2,\gamma_1,\delta_1} \otimes \cdots \otimes Q^{i_{x_6}}_{\zeta_1,\zeta_2} \otimes C_4^{\delta_2,\epsilon_2,\zeta_2}$. Note that different orderings for the contraction are possible, only constrained by the topology of the TN, where the ordering can be optimized for added computational efficiency (Pfeifer et al., 2014), although for any of them the quantum variables take both of their possible binary states and yield the same result when projected onto the satisfiable space.

The benefit of handling low-rank tensors is clear when finding the number of satisfiable configurations only ($\#SAT_{NET}$), which also contains the number of reliable configurations $\#REL_{AT}$. It is still remarkable that counting the number of configurations can be done efficiently via tensor contractions, as this information bounds the reliability problem. If the analyst wants to write the Boolean state or list the specific satisfiable configurations, one must find each of the amplitude coefficients $t^{i_{x_1},\dots,i_{x_n}}$ of the quantum Boolean state $|\Psi_{f_P}\rangle = t^{0,0,0,0,0,0}\,|000000\rangle + t^{1,0,0,0,0,0}\,|100000\rangle + t^{0,1,0,0,0,0}\,|010000\rangle + \cdots + t^{1,1,1,1,0,1}\,|111101\rangle + t^{1,1,1,1,1,0}\,|111110\rangle + t^{1,1,1,1,1,1}\,|111111\rangle$, reverting the entire enterprise to remain exponentially hard. For completeness, the individual unnormalized coefficients of the Boolean state could be obtained in general from the following inner product:

$$t^{i_{x_1},\dots,i_{x_n}} = \left\langle t^{i_{x_1},\dots,i_{x_n}} \middle| \Psi_{f_P} \right\rangle = \mathrm{Tr}[Q^{i_{x_1}}_{\alpha_1,\alpha_2} \otimes \cdots \otimes Q^{i_{x_n}}_{\omega_1,\omega_2} \otimes C_1^{\alpha_1,\beta_1,\gamma_1} \cdots \otimes C_m^{\chi_2,\psi_2,\omega_2}] \tag{6}$$

where the Tr operation sums over the shared ancillary indices of the tensor products. This TN is closed, in the sense that its full contraction yields a scalar. Such a number corresponds to the unnormalized amplitude coefficient $t^{i_{x_1},\dots,i_{x_n}} \in \{0,1\}$ of the quantum Boolean state that flags the satisfiable configurations for SAT_{NET}.

5 Ideal Networks and their Reliable States

The procedure to link SAT problems to TNs, where the SAT_{NET} formula also encodes reliability, merits additional analysis. Hence, we first explore scaling aspects of the 3SAT-based networks. To start, the 4-node planar network in Figure 5 has $H^{P_4}_{3SAT} = 41$ satisfiable configurations out of 64 possibilities. If one finds the number of reliable configurations within an ATR context, one finds $H^{P_4}_{ATR} = 38$ reliable configurations, where $H^{P_4}_{ATR} \leq H^{P_4}_{3SAT}$. These reliable configurations are found by modifying selective recursive decomposition algorithms (SRDA) for system reliability assessment (Lim and Song, 2012), where we search now for minimum spanning trees as opposed to paths as typically done (Paredes-Toro and Dueñas-Osorio, 2017)—this way ATR configurations are always found. Extending the previous network to the next possible planar cubic configuration (Figure 6), $H^{P_8}_{3SAT} = 1\,681$, whereas $H^{P_8}_{ATR} = 868$.

Two points are critical. First, counting the number of satisfiable assignments via a quantum Boolean state represented by the tensor network with elements defined as in Equations 4 and 5 requires a contraction over shared indices across ($m = 8 + n = 12$) tensors via tensor products of low rank tensors, which makes the process efficient. This is in contrast to a naive tensor product of tensors with maximum dimension for Q of $2^{2\times 12} = 16\,777\,216$ and for C of $2^{3\times 8} = 16\,777\,216$ (where the 2 in the exponent for Q reflects the COPY tensor requirement, and the 3 in the exponent for C indicates the clause or node degree). Second, the set of satisfied 3SAT configurations will always contain the set of ATR configurations (as noted in Observation 1), thus offering an upper bound to quantify system reliability—the quality of the approximation is explored formally at the end of this section. To see the bounding trends, Table 1 summarizes the ratio of the number of configurations between the satisfiability problem (#3SAT) and the all-terminal connectivity reliability ($\#REL_{AT}$) for various cubic networks of increasing size (by inserting cubic graph modules, such as the one in Figure 5, into the chain of Figure 6). Here, the trends in the number of configurations offer a worst case view into connectivity reliability, as all configurations have the same importance, or equivalently, as the probability of failure for all edges is $p_{f_e} = 0.50$—more realistic cases with smaller p_{f_e} result in smaller errors, but we report here worst cases which are easily provable via counting. Generalizing this trend, an upper bound on the number of ATR configurations stems from the 3SAT count, which grows as an exponential function of the logarithm of a function of n or $log[g(n)]$ relative to its reliability counterpart, while the space of configurations of the expanding cubic networks is more challenging as it increases exponentially as a function of n. In the general case of large $n-$sized systems, the function $g(n)$ turns exponential, but at a smaller growth rate than that of 2^n, which makes the over-counting tend to zero relative to the 2^n state space as $n \to \infty$.

Table 1: Comparison of 3SAT and SRDA satisfiable configurations for a class of cubic networks

Case	# Clauses	# Variables	# Configurations	$\#3SAT_{NET}/\#REL_{AT}$
1	4	6	64	1.079
2	8	12	4 096	1.937
3	12	18	262 144	4.089
4	16	24	16 777 216	9.362
5	20	30	1 073 741 824	22.636

A formal proposition and its proof in relation to the previous observations can be stated as follows:

Proposition 1. *The number of network configurations that satisfy the all-terminal reliability (ATR) problem, $\#REL_{AT}$, is efficiently and exactly upper-bounded by the satisfiability count, $\#SAT_{NET}$, of a CNF formula on quantum tensor networks of real networks without hyper-edges and low tree-width. This over-counting tends to zero relative to the state space of the system (as the size of the network $n \to \infty$), but it is exponential relative to $\#REL_{AT}$.*

Proof. *The cardinality of the state space of system configurations with stochastic edges grows expo-*

nentially as 2^n, with n as the number of variables or edges. The number of satisfiable configurations for the ATR problem $\#REL_{AT}$ is a subset of 2^n, starting with subsets in which at least $(m-1)$ edges are present and form a spanning tree, where m is the number of clauses or nodes. The number of satisfiable configurations for the network satistifability problem $\#SAT_{NET}$ is also a subset of 2^n, but larger than that of $\#REL_{AT}$, as it starts with subsets in which at least $(m/2)$ edges are present, so that clauses are satisfied pairwise as noted in Observation 1. However, the cardinalities of the satisfiable sets between the desired $\#REL_{AT}$ and the quantum-inspired computable $\#SAT_{NET}$ grow slower than the cardinality of the 2^n state space of the system. In our case, the excess number of configurations in $\#SAT_{NET}$ is determined by a sum of a subset of binomial coefficient expansions $T^n_{m/2,m-1}$ as follows:

$$T^n_{m/2,m-1} = \sum_{i=m/2}^{m-1} a_i \frac{n!}{i!(n-i)!} < 2^n \tag{7}$$

where the inequality is strict, and the partial sum of binomial expansion terms with respective coefficients $a_i \leq 1$ guarantees the asymptotic ratio of $T^n_{m/2,m-1}/2^n \rightarrow 0$ as $n \rightarrow \infty$. Note that the largest term of the binomial coefficient, the one choosing $(n/2)$ out of n options, always lands within the partial sum $T^n_{m/2,m-1}$, but it is a fraction a_i of it, which makes the partial sum grow slower than the 2^n growth of the system configurations space; however, such a growth will still be exponentially larger than the number of configurations in $\#REL_{AT}$. □

Establishing SAT-based CNF formulas for general networks which directly capture reliability in a TN set up is the topic of future work. However, this paper demonstrates that a quantum-inspired approach could assist with reliability problems, especially as TNs for engineering networks without hyper-edges and low-tree-width end up representable as low-rank tensor products (as edge variables generate tensors independent of the size of the network and their connectivity is local, thus also independent of the size of the network). Note that as the state space of networks grows exponentially, even for relatively small system sizes n, it becomes impractical to enumerate configurations for reliability assessment in classical computation. The quantum-inspired approach outlined here will require n qubits to cope with a superposition state that contains all satisfiable configurations of a SAT-based CNF for networks, even if only to upper-bound ATR via TNs that simulate the quantum computation. Recall that the tensor network contractions allow for low-rank tensor manipulation of large state spaces, requiring $(n+m)$ contractions in computational cost, as opposed to one contraction of dimension 2^n.

To further evaluate the ability of our quantum-inspired TN approach to count satisfiable configurations in networks, Figures 7 and 8 show computation time and the probability of satisfiability, which corresponds to ATR based on $\#REL_{AT}$ or logic satisfiability based on $\#SAT_{NET}$ for two-dimensional lattices (grids) of increasing side size. The grids have edges with $p_{f_e} = 0.50$, so that reliability or satisfiability is directly assessed by the number of satisfiable configurations. This challenging grid problem shows that state-of-the-art solvers that use binary decision diagrams (BDDs) to exactly assess $\#SAT_{NET}$ do not scale well time-wise. Approximate model counting (ApproxMC) using state-of-the-art solvers for general SAT problems are also used here[1], but for approximately counting configurations (Ivrii et al., 2016), showing competitive results, but theoretically they only guarantee outcomes with 20% error and 20% confidence—tighter errors or better confidence render the problem intractable for these grids. However, the exact TN counting outperforms other methods in time requirements with a crude implementation of tensor products, thus enabling our $\#SAT_{NET}$ counting to upper-bound $\#REL_{AT}$ in practice, particularly when actual reliability computations become harder to afford. Note that the time to compute the desired all-terminal reliability (as a function of $\#REL_{AT}$), grows exponentially when combining state-of-the-art recursive decompositions (Lim and Song, 2012; Paredes-Toro and Dueñas-Osorio, 2017) along with optimal Monte Carlo (OMCS) (Dagum et al., 2000), so as to guarantee a priori an error of maximum 20% with a confidence of 80%.

As for the probabilities of satisfying the problems at hand (Figure 8), the upper lines show results based on the SAT_{NET} problem, as a function of the count $\#SAT_{NET}$ and $p_{f_e} = 0.5$. The exact TN method is tracked by the approximate counting (ApproxMC) and not by exact BDDs as they become infeasible computationally. Despite the lax confidence interval set a priori for ApproxMC, the experiments remain remarkably close—future research will be devoted to unravel the structure of exclusive-or (XOR) constraints that may render the ApproxMC problem competitive. However, ApproxMC's results are based on poor error and confidence guarantees that cannot match the fast and exact count of our TN approach, which remains clearly competitive for a hard bound, albeit one that only upper-bounds $\#REL_{AT}$ as shown relative to OMCS.

[1] https://www.msoos.org/cryptominisat4/

The algorithmic alternatives presented here, along with the possibility of not only classical simulation, but actual computation on quantum machines in the future, open up opportunities for studying multi-state complex systems. Infrastructure systems, and their abstractions, may be suitable for such modeling approaches, where reliability is briefly explored here. However, grander reliability concepts and more computationally demanding performance measures, such as resilience, may be probed through the lens of computational logic and associated exact or approximate algorithms in the future. The connection presented here between quantum logic-based counting and reliability, is but a first step towards the more general and challenging reliability problem that requires finding the functions $f(\mathbf{x})$ for reliability explicitly.

6 Conclusions

This study provides a quantum-inspired computing perspective for system reliability bounds as a precursor for more complex applications, including general infrastructure reliability explicitly and resilience. For now, while the problem of general stochastic network reliability assessment remains unsolved, this study does link concepts from Boolean logic, quantum tensor networks (TNs), and network reliability to enable efficient computation of an exact upper-bound for particular engineering network reliability problems. We outline a process to evaluate the number of configurations that contribute to all-terminal network reliability by casting the reliability problem as a Boolean formula in conjunctive normal form (CNF) as typically done for logic-based satisfiability (SAT) problems. Such a formula, which is representable as a network for a graph satisfiability problem or SAT_{NET}, can be endowed with quibits, such that TN contractions on it unravel the number of satisfiable configurations that are encoded in a superposed quantum Boolean state. The presented process also specifies what the tensors should be to algebraically count satisfiable configurations and obtain ($\#SAT_{NET}$), which includes all ATR configurations counted in $\#REL_{AT}$.

While the quantum Boolean states from TN representations of SAT_{NET} formulas operate for networks without hyper-edges and low tree-width to upper-bound the number of REL_{AT} configurations, knowledge of the reliability polynomial will be required to count satisfiable reliability assignements directly, which is not always available and in some cases tantamount to solving the reliability problem in the first place. There are initial steps to ease this challenge in the future, especially in relation to generating SAT_{REL} formulas (explicitly capturing reliability formulas), while satisfying connectivity constraints pertinent to the network at hand, so as to approximately count with error guarantees in polynomial time via strongly probably approximately correct (SPAC) approaches (Ivrii et al., 2016; Dueñas-Osorio et al., 2017). In the meantime, we explore ideal cubic networks and two-dimensional lattices (grids), which capture SAT_{NET} problems whose formulas can be established trivially and the structure of the tensors can also be defined a priori. The main observation in regards to challenging grids is that as their size increase, most available methods that offer tight error guarantees and high confidence based on counting fail to scale for both the all-terminal reliability $\#REL_{AT}$ and the network satistiability $\#SAT_{NET}$ problems. However, our quantum-inspired TN contraction to upper-bound reliability remains computationally a linear function of the number of nodes and links of the network so long they do not contain hyper-edges and do not have wide tree-width—restrictions satisfied by most infrastructure network topologies. Overall, while quantum computers materialize, TNs allow us to efficiently simulate quantum systems by the use of classical low-rank tensor operations in lieu of uninformed 2^n tensor products.

Issues to tackle in the future are many, but primarily should focus on establishing the SAT_{REL} formulas for general reliability problems, exploiting the structure of the networks, especially their tree widths, and exploring quantum walks and quantum Monte Carlo, as such methods could be develop with theoretical guarantees on the quality of their approximations. In particular, there is room to study probabilities of events or network configurations that contribute to the tails of the distributions of the quantity of interest (Rojo, 2013). Also, as quantum computers materialize physically in the future, implementation of the counting and related algorithms is imperative (Nielsen and Chang, 2010). Alternatives also include counting satisfiable configurations via searching, where the search problem is now known to be quadratically more efficient with quantum algorithms relative to classical ones (Grover, 1997), while additive approximations can be found in quantum polynomial time (Arad and Landau, 2010). Current technologies point at tens of qubits as feasible future developments, which would provide sufficient capability to start validating the counting of reliable configurations on complex engineered networked systems. And before these quantum machines become available, the entire algebraic and combinatoric structure of the network reliability problem deserves further re-investigation (Colbourn, 1987; Shier, 1991).

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References

Arad, I., and Z. Landau, (2010). "Quantum Computation and the Evaluation of Tensor Networks." *SIAM Journal on Computing*, 39(7): 30893121. doi:10.1137/080739379.

Barker, K., J. E. Ramirez-Marquez, and C. M. Rocco, (2013). "Resilience-Based Network Component Importance Measures." *Reliability Engineering & System Safety*, 117 (September): 8997. doi:10.1016/j.ress.2013.03.012.

Biamonte, J. D., S. R. Clark, and D. Jaksch, (2011). "Categorical Tensor Network States." *AIP Advances*, 1(4): 42172. doi:10.1063/1.3672009.

Biamonte, J. D., J. Morton, and J. W. Turner, (2015). "Tensor Network Contractions for #SAT." *Journal of Statistical Physics* 160(5): 1389-1404. doi:10.1007/s10955-015-1276-z.

Bocchini, P., D. Frangopol, T. Ummenhofer, and T. Zinke, (2014). "Resilience and Sustainability of Civil Infrastructure: Toward a Unified Approach." *Journal of Infrastructure Systems*, 20(2): 4014004. doi:10.1061/(ASCE)IS.1943-555X.0000177.

Bruneau, M., S. E. Chang, R. T. Eguchi, G. C. Lee, T. D. O'Rourke, A. M. Reinhorn, M. Shinozuka, K. Tierney, W. A. Wallace, and D. von Winterfeldt, (2003). "A Framework to Quantitatively Assess and Enhance the Seismic Resilience of Communities." *Earthquake Spectra*, 19(4): 733-52. doi:10.1193/1.1623497.

Ching, J., and W.-C. Hsu, (2007). "An Efficient Method for Evaluating Origin-Destination Connectivity Reliability of Real-World Lifeline Networks." *Computer-Aided Civil and Infrastructure Engineering*, 22(8): 584-96. doi:10.1111/j.1467-8667.2007.00501.x.

Cimellaro, G. P., L. Dueñas-Osorio, and A. M. Reinhorn, (2016). "Special Issue on Resilience-Based Analysis and Design of Structures and Infrastructure Systems." *Journal of Structural Engineering* 0(0): C2016001. doi:10.1061/(ASCE)ST.1943-541X.0001592.

Colbourn, C. J., (1987). *The Combinatorics of Network Reliability*, Oxford University Press, Inc.

Crama, Y., and P. L. Hammer, (2011). *Boolean Functions: Theory, Algorithms, and Applications.* Cambridge University Press.

Dagum, P., R. Karp, M. Luby, and S. Ross, (2000). "An Optimal Algorithm for Monte Carlo Estimation."' *SIAM Journal on Computing*, 29 (5): 148496. doi:10.1137/S0097539797315306.

Daly, M. S., and C. Alexopoulos, (2006). "State-Space Partition Techniques for Multiterminal Flows in Stochastic Networks." *(*Networks), 48(2): 90-111. doi:10.1002/net.20123.

Dueñas-Osorio, L., and J. Rojo, (2011). "Reliability Assessment of Lifeline Systems with Radial Topology." *Computer-Aided Civil and Infrastructure Engineering*, 26(2): 111-28. doi:10.1111/j.1467-8667.2010.00661.x.

Dueñas-Osorio, L., K. Meel, R. Paredes, and M. Vardi, (2017). "Counting-Based Reliability Estimation for Power-Transmission Grids." *Proceedings of the Association for the Advancement of Artificial Intelligence (AAAI).*

Fischer, E., J. A. Makowsky, and E. V. Ravve, (2008). "Counting Truth Assignments of Formulas of Bounded Tree-Width or Clique-Width." *Discrete Applied Mathematics*, 156(4): 51129. doi:10.1016/j.dam.2006.06.020.

Garcia-Saez, A., and J. I. Latorre, (2012). "An Exact Tensor Network for the 3SAT Problem." *Quantum Information & Computation* 12(3-4): 283-92.

Gehl, P., and D. D'Ayala, (2016). "Development of Bayesian Networks for the Multi-Hazard Fragility Assessment of Bridge Systems." *Structural Safety*, 60(May): 37-46. doi:10.1016/j.strusafe.2016.01.006.

Grover, L. K, (1997). "Quantum mechanics helps in searching for a needle in a haystack." *Physical Review Letters*, 79(2): 32528. doi:10.1103/PhysRevLett.79.325.

Hornibrook, J.M., J.I. Colless, I.D. Conway Lamb, S.J. Pauka, H. Lu, A.C. Gossard, J.D. Watson, et al., (2015). "Cryogenic Control Architecture for Large-Scale Quantum Computing." *Physical Review Applied*, 3(2): 24010. doi:10.1103/PhysRevApplied.3.024010.

Ivrii, A., S. Malik, K. S. Meel, and M. Y. Vardi, (2016). "On Computing Minimal Independent Support and Its Applications to Sampling and Counting." *Constraints*, 21(1): 41-58. doi:10.1007/s10601-015-9204-z.

Johnson, T. H., J. D. Biamonte, S. R. Clark, and D. Jaksch, (2013). "Solving Search Problems by Strongly Simulating Quantum Circuits." *Scientific Reports*, 3(February): 1235. doi:10.1038/srep01235.

Kim, Y., and W.-H. Kang, (2013). "Network Reliability Analysis of Complex Systems Using a Non-Simulation-Based Method." *Reliability Engineering & System Safety*, 110(February): 80-88. doi:10.1016/j.ress.2012.09.012.

Landsberg, J. M, (2011). *Tensors: Geometry and Applications. American Mathematical Society.*

Li, J., and J. He, (2002). "A recursive decomposition algorithm for network seismic reliability evaluation". *Earthquake Engineering and Structural Dynamics*, 31: 1525-1539.

Lim, H.-W., and J. Song, (2012). "Efficient Risk Assessment of Lifeline Networks under Spatially Correlated Ground Motions Using Selective Recursive Decomposition Algorithm." *Earthquake Engineering & Structural Dynamics*, 41(13): 186182. doi:10.1002/eqe.2162.

Malik, S., and L. Zhang, (2009). "Boolean Satisfiability from Theoretical Hardness to Practical Success." *Commun. ACM*, 52(8): 76-82.

Markov, I. L., and Y. Shi, (2008). "Simulating Quantum Computation by Contracting Tensor Networks." *SIAM Journal on Computing*, 38(3): 963-81.

Morton, J., and J. Biamonte, (2012). "Undecidability in Tensor Network States." *Physical Review A*, 86(3): 30301. doi:10.1103/PhysRevA.86.030301.

Murg, V., F. Verstraete, O. Legeza, and R. M. Noack, (2010). "Simulating Strongly Correlated Quantum Systems with Tree Tensor Networks." *Physical Review B*, 82(20): 205105. doi:10.1103/PhysRevB.82.205105.

Nielsen, M. A., and I. L. Chuang, (2010). *Quantum Computation and Quantum Information*, Cambridge University Press.

Orus, R., (2014). "A Practical Introduction to Tensor Networks: Matrix Product States and Projected Entangled Pair States." *Annals of Physics*, 349(October): 117-58. doi:10.1016/j.aop.2014.06.013.

Paredes-Toro, R. and L. Dueñas-Osorio, (2017). "Computing Seismic Risk in Interdependent Lifeline Systems." *Earthquake Engineering and Structural Dynamics*, in review.

Pfeifer, R. N. C., J. Haegeman, and F. Verstraete, (2014). "Faster Identification of Optimal Contraction Sequences for Tensor Networks." *Physical Review E*, 90(3). doi:10.1103/PhysRevE.90.033315.

Provan, J., and M. Ball, (1983). "The Complexity of Counting Cuts and of Computing the Probability That a Graph Is Connected." *SIAM Journal on Computing*, 12(4): 777-88. doi:10.1137/0212053.

Rieffel, E. G., and W. Polak, (2014). *Quantum Computing: A Gentle Introduction.* MIT Press.

Rojo, J., (2013). "Heavy-Tailed Densities." *Wiley Interdisciplinary Reviews (WIREs): Computational Statistics* 5(1): 3040. doi:10.1002/wics.1236.

Samer, M., and S. Szeider, (2010). "Algorithms for Propositional Model Counting." *Journal of Discrete Algorithms*, 8(1): 50-64. doi:10.1016/j.jda.2009.06.002.

Shier, D. R, (1991). *Network Reliability and Algebraic Structures*, Clarendon Press.

Song, J., and W.-H. Kang, (2009). "System reliability and sensitivity under statistical dependence by matrix-based system reliability method". *Structural Safety*, 31: 148-156.

Stern, R. E., (2015). *Improved methods for fast system reliability analysis through machine-learning-based surrogate models.* Master of Science thesis, University of Illinois, Urbana-Champaign.

Straub, D., and A. Der Kiureghian, (2010). “Bayesian Network Enhanced with Structural Reliability Methods: Methodology.” *Journal of Engineering Mechanics*, 136(10): 1248-58. doi:10.1061/(ASCE)EM.1943-7889.0000173.

Susskind, L., and A. Friedman, (2014). *Quantum Mechanics: The Theoretical Minimum.* Basic Books.

Valiron, B., N. J. Ross, P. Selinger, D. S. Alexander, and J. M. Smith, (2015). “Programming the Quantum Future.” *Communications of the ACM*, 58(8): 5261. doi:10.1145/2699415.

van Meter, R., and C. Horsman, (2013). “A Blueprint for Building a Quantum Computer.” *Communications of the ACM*, 56(10): 8493. doi:10.1145/2494568.

Yoran, N., and A. J. Short. 2006. “Classical Simulation of Limited-Width Cluster-State Quantum Computation.” *Physical Review Letters*, 96(17): 170503. doi:10.1103/PhysRevLett.96.170503.

Zuev, K. M., S. Wu, and J. L. Beck, (2015). “General Network Reliability Problem and Its Efficient Solution by Subset Simulation.” *Probabilistic Engineering Mechanics*, 40(April): 25-35. doi:10.1016/j.probengmech.2015.02.002.

List of Figures

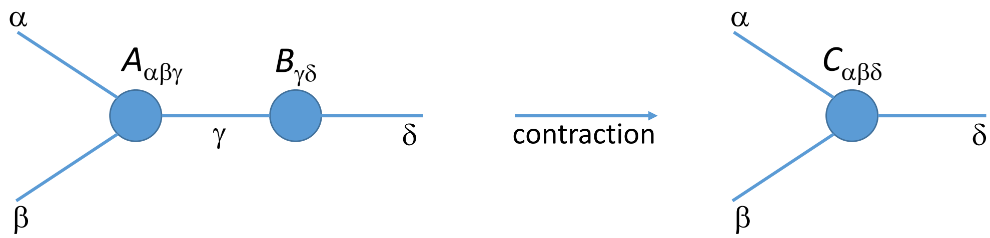

Figure 1: Tensor network contraction via index γ.

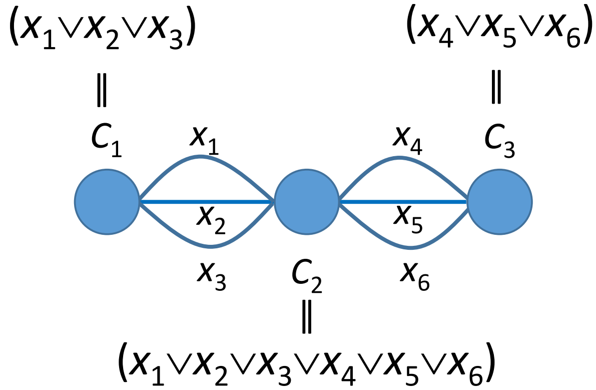

Figure 2: Network representation of a simple CNF formula with 6 variables (edges) and 3 clauses (nodes).

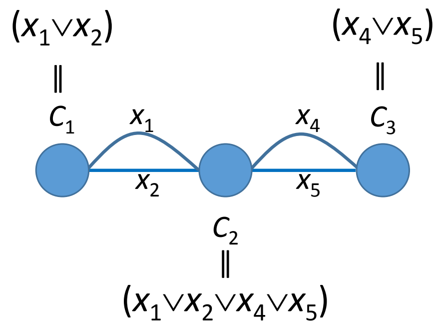

Figure 3: Network representation of a CNF formula for the simpler version of the series-parallel system in Figure 2.

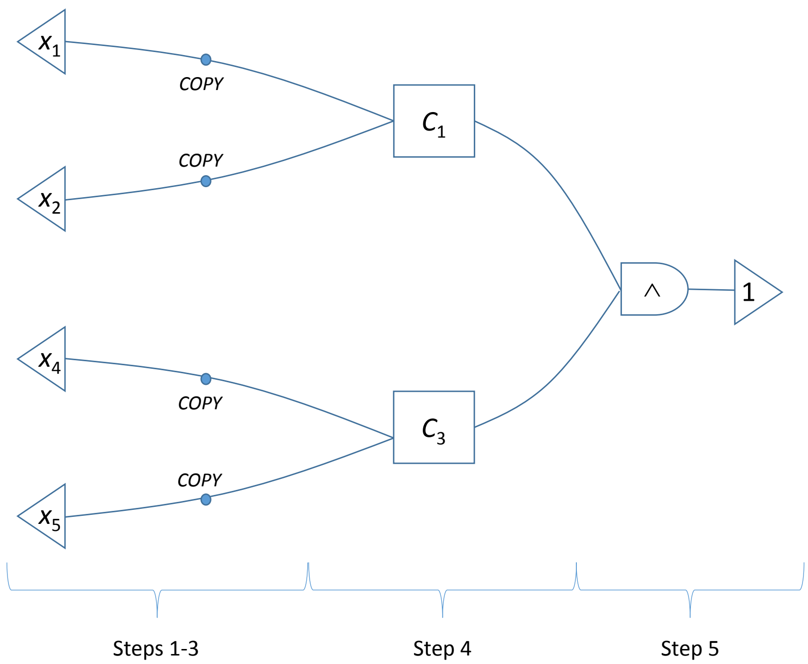

Figure 4: Process to evaluate the TN of the quantum Boolean state for the SAT formula encoded in Figure 3.

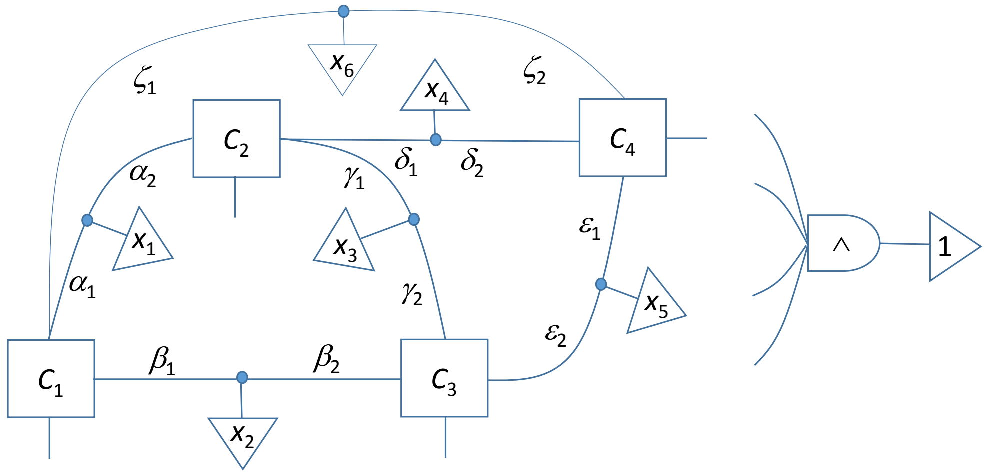

Figure 5: Maximally planar tensor network on 6 variables and 4 clauses. Each clause connects to the AND operator that projects onto the satisfied configuration $|1\rangle$

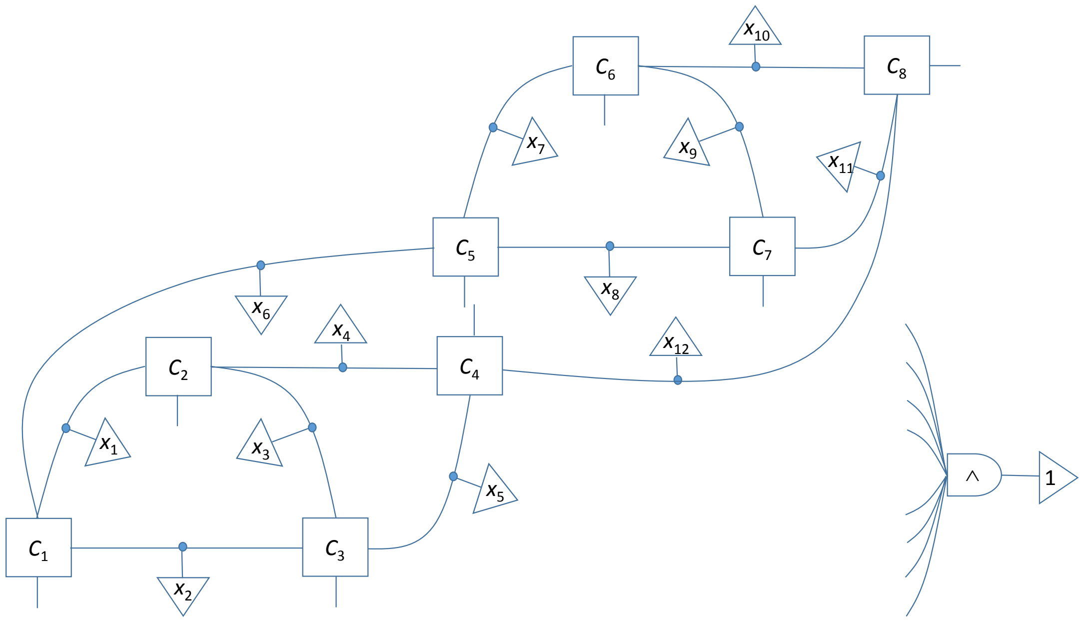

Figure 6: Maximally planar tensor network on 12 variables and 8 clauses. Each clause connects to the AND operator that projects onto $|1\rangle$

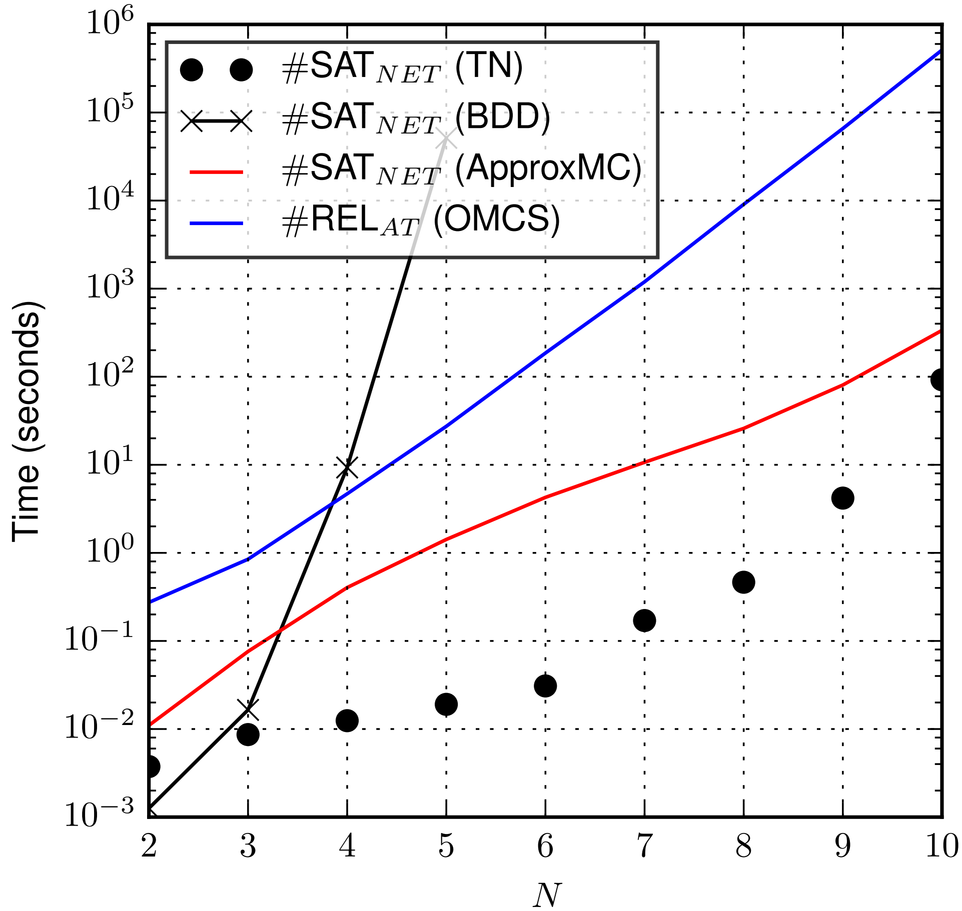

Figure 7: Time scaling of counting-based all-terminal reliability and network satisfiability methods

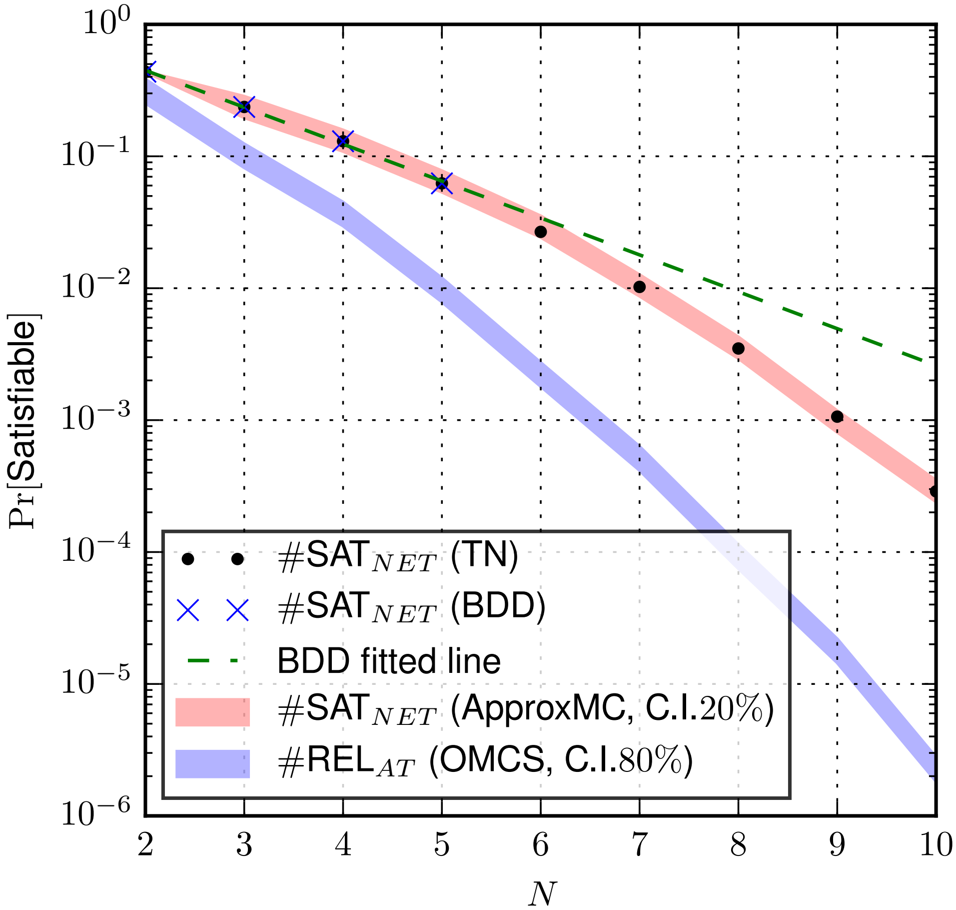

Figure 8: Counting-based all-terminal reliability and network satisfiability probability with guaranteed assessments for $p_{f_e} = 0.50$

Highlights

Quantum-Inspired Boolean States for Bounding Engineering Network Reliability Assessment

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- Tensor networks (TNs) enable novel algebraic reliability assessments
- Quantum Boolean states on TNs bound reliable network configurations
- Satisfiability (SAT) formulas inform tensor contractions for reliability counting
- Low-rank tensor contractions efficiently count by simulating quantum systems
- Strategies for bounding SAT_{NET} counts can inform general reliability assessment