nature physics

REVIEW ARTICLE

PUBLISHED ONLINE: 1 JULY 2015 | DOI: 10.1038/NPHYS3378

Anomalous critical and supercritical phenomena in explosive percolation

Raissa M. D'Souza[1] and Jan Nagler[2]

The emergence of large-scale connectivity on an underlying network or lattice, the so-called percolation transition, has a profound impact on the system's macroscopic behaviours. There is thus great interest in controlling the location of the percolation transition to either enhance or delay its onset and, more generally, in understanding the consequences of such control interventions. Here we review explosive percolation, the sudden emergence of large-scale connectivity that results from repeated, small interventions designed to delay the percolation transition. These transitions exhibit drastic, unanticipated and exciting consequences that make explosive percolation an emerging paradigm for modelling real-world systems ranging from social networks to nanotubes.

The percolation transition, named for the prototypical mathematical problem of pouring liquid through a porous material, describes the onset of large-scale connectivity on an underlying network or lattice. At times, ensuring large-scale connectivity is essential: a transportation network (such as the world-wide airline network) or a communication system (such the Internet) is useful only if a large fraction of the nodes are connected. Yet, in other contexts, large-scale connectivity is a liability: under certain conditions, a virus spreading on a well-connected social or computer network can reach enough nodes to cause an epidemic. Thus, percolation theory is a theoretical underpinning across a range of fields[1,2] and the desire to enhance or delay the onset of percolation has been of interest for many years. The consequences of delaying the transition have only recently been established and here we review explosive percolation (EP), the phenomenon that usually results from repeated, small interventions designed to delay the percolation transition. The onset can indeed be significantly delayed, but once the percolation transition is inevitably reached, large-scale connectivity emerges suddenly.

The traditional approach for constructing a random graph, the Erdős–Rényi model, considers a collection of N isolated nodes, with each possible edge between two distinct nodes added to the graph with probability p (refs 3–5). This is a static formulation with no dependence on the history of how edges have been added to the graph. A mathematically equivalent kinetic formulation is initialized with N originally isolated nodes with a randomly sampled edge added at each discrete time step[6]. Letting T denote the number of steps, the process is parameterized by the relative number of introduced edges $t = T/N$, and typically analysed in the thermodynamic limit of infinite size N. Below some critical $t = t_c$ the resulting graph is disjoint, consisting of small isolated clusters (or components) of connected nodes. (See Fig. 1c for an illustration of distinct components.) Let C denote the largest component and $|C|$ its size. For the Erdős–Rényi model, the order parameter $|C|$ undergoes a second-order transition at $t_c = 1/2$ where, below t_c, $|C|$ is logarithmic in N and, above t_c, there is a unique largest component with size that grows linearly with N (ref. 7).

The impact of choice

At a Fields Institute workshop in 2000, Dimitris Achlioptas introduced an extension to this standard process, designed to enhance or delay the percolation transition based on the 'power of two choices' as used in randomized algorithms[8–11]. Starting with N isolated nodes, rather than choosing one edge in each discrete time step, choose two candidate edges, denoted $\{e_1, e_2\}$, and examine the consequence of adding each one individually to the graph. The edge that best satisfies a set of pre-determined selection criteria is added to the graph and the second candidate edge is discarded for this time step. Selection criteria can include keeping components small (delaying percolation), or growing a large component as quickly as possible (enhancing percolation). The process can also be generalized to consider $m \geq 2$ candidate edges at each time, where m is kept constant. Such an 'm-edge' competitive graph-evolution algorithm has come to be known as an Achlioptas process.

Achlioptas processes were first analysed by Tom Bohman and Alan Frieze[12] in the context of 'bounded-size' rules, where all components of size K or greater are treated equivalently. For the Bohman and Frieze (BF) process, e_1 is accepted if it joins two isolated nodes (and e_2 rejected), otherwise e_2 is accepted (and e_1 rejected). Thus, only components of size one (isolated nodes) are distinguished, and all components of size $K \geq 2$ are treated equivalently. A rigorous proof shows that BF delays the percolation transition when compared to ER, but the nature of the transition has not been investigated[12].

BF can be modelled as a cluster-aggregation process based on the Smoluchowski coagulation equation[13,14]. This assumes that, at each discrete time step, two independent components are merged, and thus implies that the maximum number of edges possible is $N-1$. The error introduced from the violation of this assumption nearing the critical point can be rigorously analysed and, as a result, it is conjectured that all bounded-size rules lead to a continuous phase transition[15]. Cluster-aggregation analysis is a technique used in many studies discussed throughout this review. Note that cluster-aggregation processes necessarily end at $t = (N-1)/N$ when only one component remains, whereas on an undirected network the maximum edge density attainable is $t = (N-1)/2$.

[1]Department of Computer Science and Department of Mechanical and Aerospace Engineering at University of California, Davis, California 95616, USA.
[2]Computational Physics for Engineering Materials, IfB, ETH Zürich, Wolfgang-Pauli-Strasse 27, CH 8093 Zürich, Switzerland.
e-mail: raissa@cse.ucdavis.edu; jnagler@ethz.ch

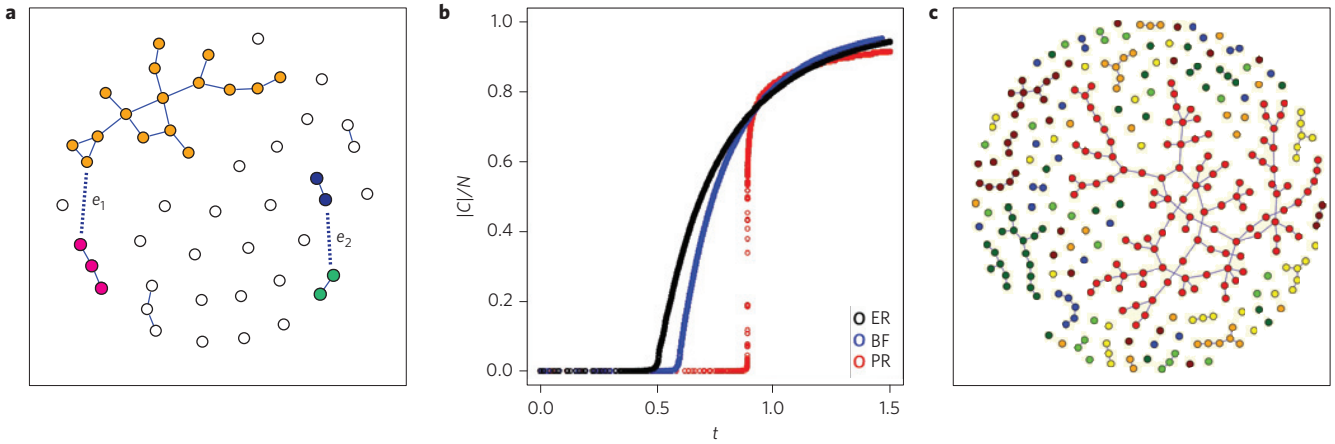

Figure 1 | Schematic of explosive percolation. a, At each time step of the product rule (PR) process, two edges, e_1 and e_2, compete for addition. Here the product of the components merged by e_1 is $3\times16=48$ and by e_2 is $2\times2=4$, so e_2 is accepted in and e_1 rejected. **b**, Typical evolution of an Erdős–Rényi (ER), Bohman Frieze (BF) and PR process on a system of size $N=10^6$. Plotted is the fractional size of the largest component, $|C|/N$, as a function of edge density t. **c**, A sample ER network in the supercritical regime, with the nodes in each distinct component rendered in the same colour. The largest component, C, is indicated in red.

Novel critical properties

Analysis of unbounded-size rules is more challenging. The first significant study appeared in 2009 (ref. 16) and focused on the product rule (PR), an Achlioptas process defined as follows. Starting from N isolated nodes, two candidate edges $\{e_1,e_2\}$ are chosen uniformly at random at each discrete time step. For $t<t_c$, the largest components are logarithmic in N and thus, with high probability, the two edges involve four distinct components with sizes respectively denoted $|C_a|$, $|C_b|$, $|C_c|$, $|C_d|$. Let e_1 denote the edge which joins the first two components, and e_2 the second two. If $|C_a|\times|C_b|<|C_c|\times|C_d|$, then e_1 is added to the graph. Otherwise, e_2 is added. In other words, we retain the edge that minimizes the product of the two components that would be joined by that edge (see Fig. 1a).

A typical realization of a product rule process is shown in Fig. 1b, together with realization of an Erdős–Rényi[3,4] and a Bohman–Frieze process[12] on a system of size $N=10^6$. Note that the onset of large-scale connectivity is considerably delayed for the product rule process, and that it emerges drastically, going from sublinear to a level approximately equal to the corresponding Erdős–Rényi and Bohman–Frieze processes during an almost imperceptible change in edge density. Our numerical simulations make use of the commonly used Newman–Ziff algorithm for efficient computation of percolation[17].

The 2009 study focused on direct simulation of the product rule process[16]. To quantify the abruptness of the transition, the scaling window as a function of system size N, denoted $\Delta_N(\gamma,A)$, was analysed. This measures the number of edges required for $|C|$ to transition from being smaller than N^γ to being larger than AN, with typical choices of parameters being $\gamma=A=1/2$. Systems up to size $N\sim6\times10^7$ were studied and the results indicated a sublinear scaling window, $\Delta_N(0.5,0.5)\propto N^{2/3}$, and $t_c\approx0.888$ (ref. 16). The associated change in edge density, $\Delta_N(0.5,0.5)/N\propto N^{-1/3}$, vanishes in the limit of large N—providing strong, yet ultimately misleading, evidence that large-scale connectivity emerges in a discontinuous phase transition.

Many further studies followed soon after ref. 16, including analysis of the product rule on a lattice[18] and on networks with power-law degree distributions[19,20]. These studies provided similar evidence for a discontinuous percolation transition. But they also highlighted the existence of scaling behaviours characteristic of second-order phase transitions[21,22]. Many other Achlioptas processes have now been analysed, such as rules using the sum rather than product and rules with $m>2$ choices[23–26]. Similar results of sublinear scaling windows and critical scaling behaviours are observed (see ref. 27 for a review of many of these processes). Note that models exhibiting a discontinuous jump in the order parameter, but diverging length scales characteristic of second-order transitions, are well established for models of 'jamming percolation' on low-dimensional lattices[28–31]. These models incorporate spatial correlations intended to capture glassy dynamics in materials.

Rather than the scaling window, the impact of a single edge[32] provides a more crisp analysis. Soon after the early studies appeared, it was shown that, for the product rule and similar m-edge processes, the maximum change in the relative size of the largest component from the addition of a single edge decays as a power law with system size, $\Delta C_{\max}\sim N^{-\beta}$ (refs 32,33). Thus the process is continuous as $N\to\infty$. The rate of decay is typically fairly small ($\beta=0.065$ for the product rule[33]), leading to large discrete jumps in systems that are orders of magnitude larger than real-world networks. More details are included later in this review with respect to applications of explosive percolation.

Mounting numerical evidence and heuristic arguments indicated that Achlioptas processes lead, in fact, to a continuous phase transition[34–37], but with a universality class distinct from any previously observed[35,37]. See ref. 27 for a review of the critical exponents found. Finally, in 2011 a rigorous proof by Oliver Riordan and Lutz Warnke showed that any Achlioptas process leads to a continuous percolation transition[38]. They proved, in essence, that the number of subcritical components that join together to form the emergent macroscopic-sized component is not sub-extensive in system size. In the words of Eric Friedman and Adam Landsberg, Achlioptas processes do not lead to the build-up of a 'powder keg'[23,39], which is a collection of components that contain cN nodes in total where the sizes of the components diverge to infinity as $N\to\infty$ for some constant c. Merging the components of such a powder keg would lead to a discontinuous percolation transition.

Yet, Riordan and Warnke showed that, for a random graph, if the number of random choices m is allowed to increase in any way with system size N, so that $m\to\infty$ as $N\to\infty$ (for example, $m\sim\log(\log N)$), then this is sufficient to allow a discontinuous transition. For rules not based on randomly chosen m node pairs, however, a discontinuous transition is not guaranteed to occur. Many explosive percolation processes with alternative mechanisms that lead to genuinely discontinuous percolation transitions have now been discovered, as will be discussed later in this review.

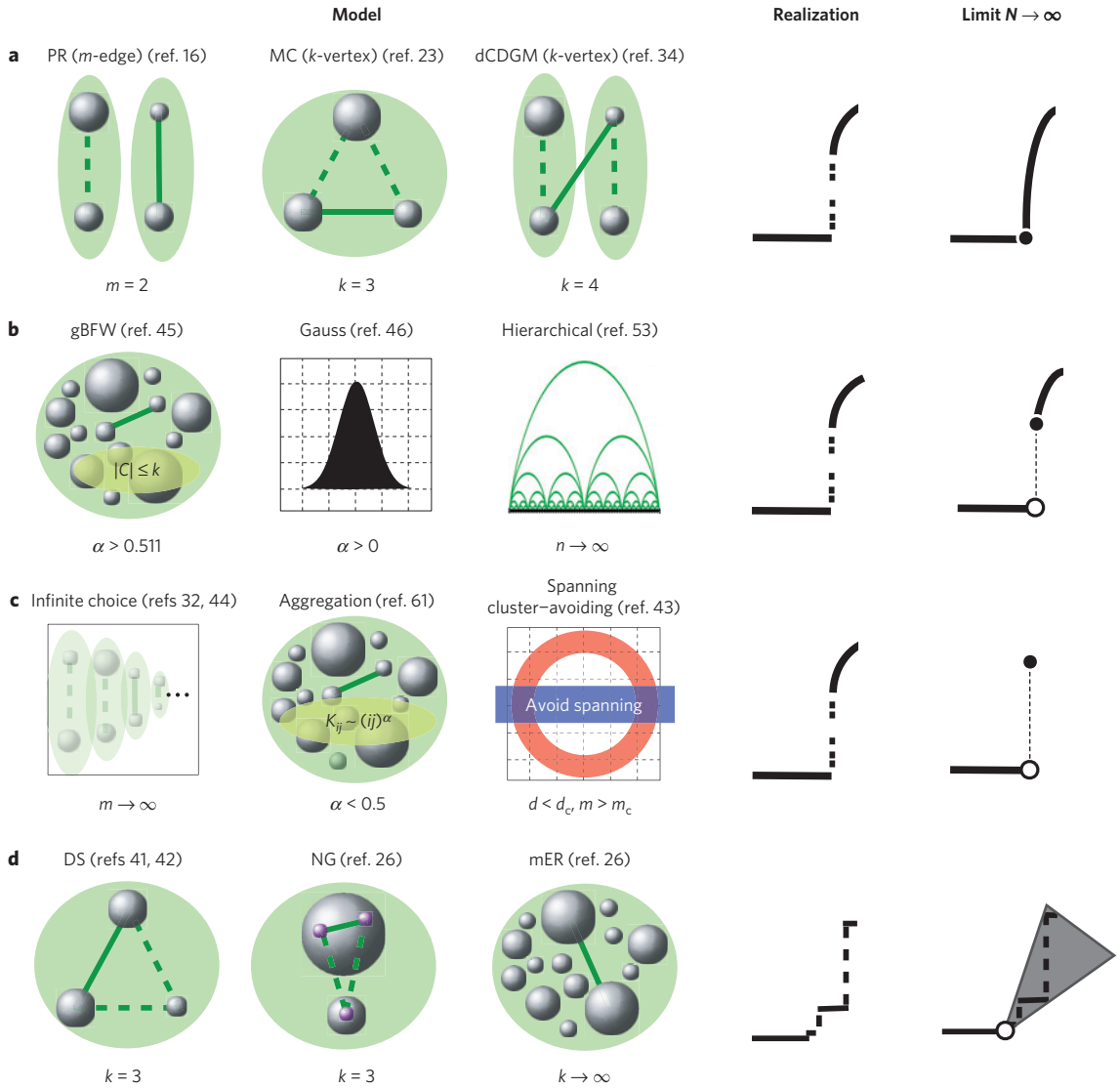

Figure 2 | Classes of explosive percolation. a, The product rule (PR), minimal cluster rule (MC) and a model named after its creators (dCDGM) are examples of explosive percolation processes that are continuous in the thermodynamic limit but nevertheless exhibit substantial jumps in the order parameter for any finite system. For m-edge rules, m links compete for addition. For k-vertex rules, all possible $k(k-1)/2$ node pairs compete. **b**, Models that exhibit a single genuine jump in the order parameter $|C|/N$ well in advance of the end of the process. The hierarchical model results from the construction of n generations of long-range bonds, in the limit of $n \to \infty$. **c**, Models that exhibit a single discontinuous jump in the order parameter $|C|/N$ at the end of the process, resulting in a 'global' jump encompassing the full system. All rules in **a**–**c** delay the onset of percolation and avoid mergers of large clusters. **d**, Non-convergent, non-self-averaging models that exhibit a staircase with genuinely discontinuous steps, including the devil's staircase (DS), Nagler–Gutch (NG) and modified Erdős–Rényi (mER) models. Even in the thermodynamic limit the staircases are stochastic (both the size of the steps and their location). For those models, mergers of large components are not explicitly suppressed.

Stochastic staircases in the supercritical regime

Achlioptas processes (or m-edge rules) are continuous. Although a finite realization may show large discrete jumps, in the limit $N \to \infty$ the evolution converges to a smooth, continuous function, as illustrated in Fig. 2a. But, remarkably, the more general class of 'k-vertex rules' (which consider a fixed number of candidate vertices rather than edges) allows new possibilities.

To understand the distinction, first consider an m-edge rule. The m vertex pairs are chosen uniformly at random. Hence, as long as there are at least two components in the system, there is a non-zero probability that all candidate edges chosen at a given step have exactly one end-point (or vertex) in the largest component C. Thus, independent of the rule, the probability, $P_{\rm gr}$, that the largest component merges with another smaller component is necessarily non-zero (and even increases during the process as C grows). This results in growth of the largest component being dominant, preventing the build-up of a powder keg, and leading to continuous growth of $|C|/N$ in the thermodynamic limit[32,40].

A different mechanism underlies k-vertex rules. The devil's staircase rule is a three-vertex rule that preferentially merges components of equal (or similar) size or adds an intra-cluster edge[41,42]. Hence, regardless of how many of the chosen vertices reside in C, it is impossible that C merges with a smaller component, meaning $P_{\rm gr} = 0$. Instead, smaller components merge together—sometimes becoming the new largest component (which can then no longer grow directly). This condition necessarily implies one or more discontinuous transitions during the process[32]. In particular, the devil's staircase rule exhibits a continuous percolation transition at t_c, yet exhibits infinitely many discontinuous jumps at $t > t_c$, with the 'first' such jump within an arbitrary vicinity of the initial

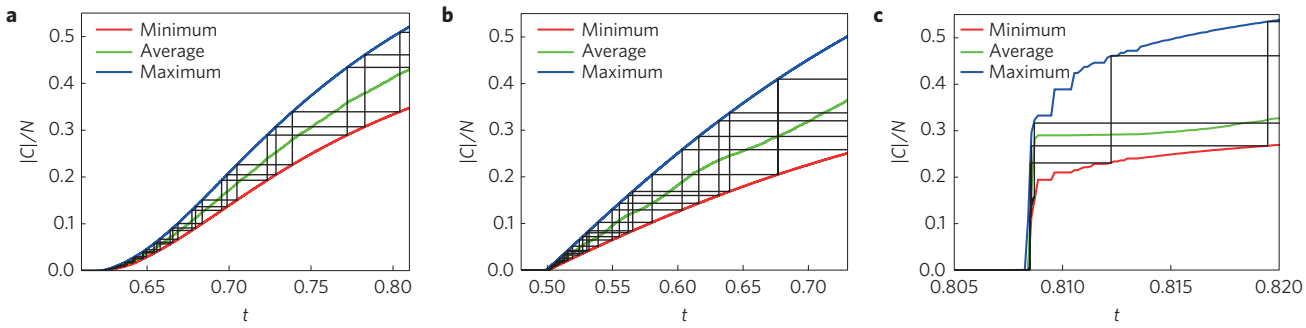

Figure 3 | Explosive percolation with stochastic staircases. a–c, Models of explosive percolation can be non-convergent and not self-averaging[26]. Shown are realizations of devil's staircases of genuinely discontinuous jumps with the relative size of the largest component $|C|/N$ as a function of t for several distinct realizations (black lines), together with ensemble average (green line), minimum (red line) and maximum (blue line). **a**, Devil's staircase model analysed in refs 41,42. **b**, Modified Erdős–Rényi model from ref. 26. **c**, Nagler–Gutch model analysed in ref. 26. The averages were obtained from 1,500 realizations for systems of size $N=2^{30}$.

percolation transition. Thus continuity at the first connectivity transition and discontinuity of the percolation process can be compatible. Moreover, the devil's staircase (a Cantor function with discrete jumps) is random, even in the thermodynamic limit, meaning that the locations of the jumps are stochastic variables, as illustrated in Fig. 3a.

Other rules where the order parameter $|C|/N$ is 'blurred' in the supercritical regime and does not converge to a function of t in the thermodynamic limit were reported in ref. 26. These models exhibit tremendous variation from one realization to another in the supercritical regime[26] (see Fig. 3b,c). This behaviour is called non-self-averaging and is quantified by the relative variance of the order parameter $R_v(C)$ over an ensemble of realizations. For continuous phase transitions it is well known that large fluctuations in $R_v(C)$ are observed only in the critical window and that they collapse to a singular peak at t_c in the thermodynamic limit. Figure 4 shows the lack of self-averaging for the devil's staircase model and a modified Erdős–Rényi model as characterized by elevated values of $R_v(C)$ in the supercritical regime. Most remarkably, large fluctuations in $R_v(C)$ can be observed even in the early, subcritical evolution, as shown in Fig. 4. These fluctuations can have predictive power, as discussed later. We have illustrated this class of explosive percolation phenomenon with genuine stochastic staircases in Fig. 2d.

Lattice models and global percolation phenomena. Unlike on a random network, an m-edge Achlioptas process on a lattice can yield a discontinuous percolation transition at t_c. Percolation on a lattice is often measured by the emergence of a spanning cluster—a path of activated links that connect sites from one side of the lattice to another. In the spanning cluster avoidance model, the emergence of a spanning cluster is discontinuous for a lattice with dimension $d < d_c = 6$ as long as $m \geq m_c = d/(d - d_{BB})$, where d_{BB} is the fractal dimension of the 'backbone' (which has been calculated analytically and measured numerically)[43]. When $d = 2$, $m_c \approx 2.554$, so setting $m = 3$ is sufficient for a discontinuous transition. Yet, an interesting distinction occurs for $m = m_c$ versus $m > m_c$. For $m = m_c$, the discontinuous percolation transition occurs at some intermediate t_c during the process, as shown in Fig. 2b for other models exhibiting this class of explosive percolation phenomenon. In contrast, for $m > m_c$ the process acts globally, so when the spanning cluster emerges, it encompasses the entire system. Such 'global' percolation also happens for an m-edge Achlioptas process on a random graph in the limit $m \to \infty$. Instead of metric or geometrical confinements, the rule has unrestricted access to the entire collection of components. There, a giant percolating component emerges only in the final step of the process when only one component remains, as first discussed in ref. 44. This class of explosive percolation phenomenon with global jumps is illustrated in Fig. 2c.

Underlying mechanisms

Erdős–Rényi percolation can be considered a form of 'multiplicative coalescence'[14]. From the kinetic perspective, at each discrete time step two vertices are chosen uniformly at random and linked by an edge. The probability that a randomly chosen vertex is in a particular component of size j is j/N. Thus, to first order, the probability that a randomly selected edge merges a particular component of size j with a particular one of size i is proportional to ij/N^2 (see ref. 14 for more rigorous details). As with gravitational attraction, the force between two bodies is proportional to the product of their masses. It suffices to say that, under Erdős–Rényi evolution, the largest components quickly merge together to form one larger component, hence amplifying the likelihood of that component being included in subsequent edges. Such arguments provide the intuition for why there is only one unique giant component in the supercritical regime for Erdős–Rényi percolation.

The appearance of ref. 16 led to increased activity in the field and to the discovery of several random graph percolation models that exhibit truly discontinuous transitions. These models break multiplicative coalescence, allowing instead a multitude of components with sizes similar to that of the largest component. This creates the necessary powder keg[23] in the subcritical regime, and can allow multiple, coexisting giant components in the supercritical regime[45].

The class of explosive percolation phenomenon with a discontinuous, but non-global, jump is illustrated in Fig. 2b. Two of the models shown[45,46] work by suppressing the growth of the largest component. In ref. 46 a regular lattice is the underlying substrate and a single edge is examined at a time ($m = 1$). If a randomly chosen edge would not increase the current size of the largest component then it is accepted. Otherwise it is rejected with a probability function that decays as a Gaussian distribution centred on the average cluster size. Thus, components that are similar in size to the average are favoured. Clear signatures of a first-order transition are observed, such as bimodal peaks for the cluster size distribution, indicating the coexistence of percolative and non-percolative states in finite systems at t_c. In contrast, the random graph version of this Gaussian model[47] exhibits a discontinuous transition at the end of the process (as illustrated in Fig. 2c).

Another model introduced by Bohman, Frieze and Wormald (BFW; ref. 48) is analysed in ref. 45. The model considers a single edge at a time. The edge is added to the graph if the resulting component would be smaller than some specified size k. Otherwise, the edge is rejected provided that a stringent lower bound on edge density is always satisfied. If the edge cannot be rejected, then the cap

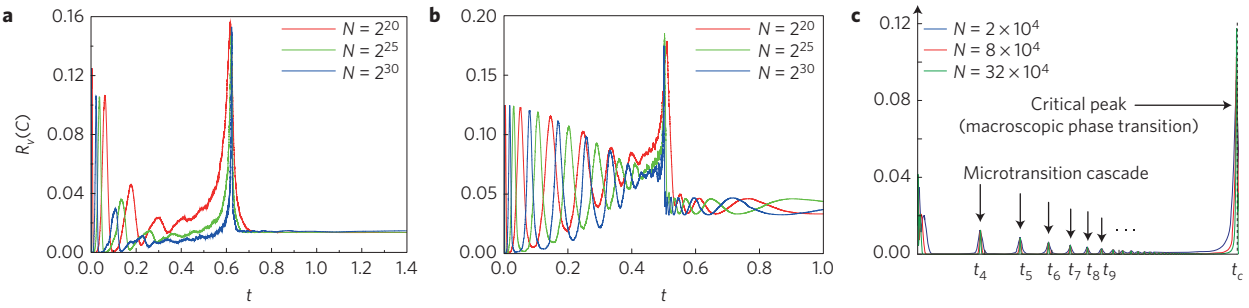

Figure 4 | Non-self-averaging in explosive percolation. a–c, Relative variance R_v of the largest component in dependence on the link density for the devil's staircase model (**a**), the modified Erdős–Rényi model (**b**) and the generalized BFW model (**c**). In the supercritical regime of the devil's staircase model (**a**) the system is non-self-averaging, characterized by extended regions of $R_v \neq 0$, for $N \to \infty$. For the modified Erdős–Rényi model (**b**), remarkably, R_v as a function of t follows intricate patterns, such as oscillations with amplitudes that seem to survive in the thermodynamic limit, both in the subcritical and supercritical regime. The generalized BFW model (**c**) studied in ref. 73 exhibits peaks in R_v at well-defined intervals that exhibit a discrete scale invariance and survive in the thermodynamic limit, and moreover predict the percolation point.

k is increased incrementally while the lower bound correspondingly decreases as a function of k until reaching an asymptotic limiting value, α. In the original model $\alpha = 1/2$ (such that asymptotically one-half of all edges must be accepted)[48]. Ref. 45 shows that this process leads to a truly discontinuous transition in which multiple giant components emerge simultaneously, as shown in Fig. 5a. In the supercritical regime, any edge leading to an increase in the cap size k can be simply rejected, and thus the multiple giant components coexist without merging.

One can generalize the BFW model by allowing α to vary, providing a parameter for tuning the number of giant components that emerge at t_c (see inset to Fig. 5a). For some values of α the coexistence is unstable, leading to additional discontinuous jumps[49,50]. The critical behaviour for $\alpha > 0.511$ when only one giant emerges is illustrated in Fig. 2b. That the maximum change in relative size from the addition of a single edge $\Delta C_{\max}$ is invariant with N for BFW, but decays as $N^{-0.065}$ for the product rule, is shown in Fig. 5b. The evolution of the component size distribution as t increases is shown in Fig. 5c, illustrating the build-up of the powder keg. Here $n(s)$ denotes the number of components of size s divided by N.

The BFW process gives rise to the simple underlying mechanism of growth by overtaking[51]. The growth of the largest component is severely limited, as it can merge only with isolated nodes. Instead, all significant changes in $|C|$ result from two smaller components merging together and overtaking the previous largest component to become the new largest component[51].

Models that have been shown to lead to a single discontinuous percolation transition on random graphs include a restricted Erdős–Rényi process, where one end-point of the edge is chosen uniformly at random and the other is chosen randomly from a restricted set[52]. Ordinary percolation on a hierarchical network can also yield a discontinuous percolation transition at some intermediate t_c during the process[53] (see Fig. 2b). Furthermore, there is a Hamiltonian formulation that connects evolution via Achlioptas processes with an equilibrium statistical mechanics process[54], highlighting the role of non-local information in discontinuous percolation. It was also shown recently that modelling cascading failure on interacting networks via percolation typically involves a discontinuous transition from global connectedness to disintegration of the network[55].

Although percolation considers the evolution of the network structure, explosive percolation has also motivated exploration of dynamical processes taking place on a fixed network structure, such as the 'explosive Ising' model[56] and 'explosive synchronization'[57]. The latter was first shown in a network of oscillators when the natural frequency of each oscillator is positively correlated with its degree. A recent study revealed that suppressing the formation of large clusters is the common mechanism underlying all now-explored models of explosive synchronization[58], linking the mechanism with explosive percolation.

Notably, there have long been models of percolation known to show discontinuous transitions, such as k-core percolation and models of jamming on low-dimensional lattices[28–31]. Mechanisms underlying these processes are primarily cooperative interactions[59] and correlated percolation[31]. Refs 31,59 include interesting discussions connecting these known models to the more recent work on explosive percolation, highlighting lattice models, generalized epidemic models and the statistical mechanics of exponential random graphs.

Explosive percolation in real-world networks

Explosive percolation in finite systems. The rigorous proof by Riordan and Warnke[38] shows that in the limit $N \to \infty$ the scaling window is linear in system size N, but numerical evidence on systems up to size $N \sim 10^7$ indicates the window is sublinear[16]. Thus, there must be a crossover length, N^*, where the system becomes large enough such that actual realizations show convergence to the asymptotic limiting behaviour. A method for estimating the crossover length is to model the expected evolution of a network using cluster-aggregation equations, such as the Smoluchowski coagulation equation[13,14], which is a mean-field analysis over the ensemble of all possible random graphs[14]. Cluster-aggregation approaches to general percolation provide useful analytical tools[6,60], which have been useful for modelling explosive percolation processes[24,33,61].

Cluster-aggregation models related to the product rule suggest that the largest component obeys a scaling relation $|C|/N \sim (t - t_c)^{0.0555}$ for t just above t_c (ref. 34). This indicates unusually rapid, albeit continuous, growth. As discussed earlier, the maximum impact from the addition of a single edge for such processes obeys the relation $\Delta C_{\max} \sim N^{-\beta}$, which for very small values of β coincides with the scaling of the largest component $|C|/N \sim (1/N)^{\beta} = (\Delta t)^{\beta}$, as the addition of a single edge corresponds to $\Delta t = 1/N$. For $t < t_c$, by definition $|C|/N \to 0$. As we pass into the critical regime $|C|/N \sim (1/N)^{\beta}$. This means that, for a system of size $N = 10^{1/\beta}$, the addition of a single edge causes the order parameter to exhibit a discrete jump equal to ten per cent of the system size, $\Delta|C|/N = 0.1$. For a process with $\beta = 0.0555 \approx 1/18$, the crossover length $N^* > 10^{18}$.

The thermodynamic limit is extremely relevant when considering phase transitions in physical materials, where system sizes are of the order of Avogadro's number, $N \sim 10^{23}$. But real-world networks, such as the Internet, the world-wide airline network, online social networks and gene interaction networks, are all considerably smaller than 10^{18}. Although fixed-choice Achlioptas processes yield

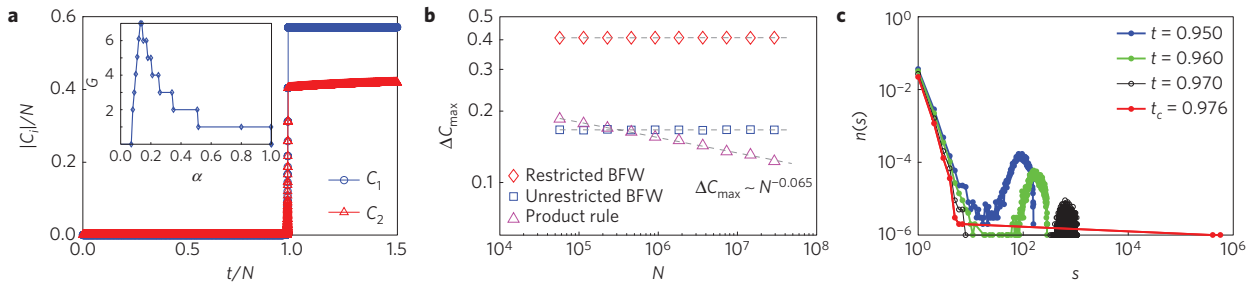

Figure 5 | Multiple giant components and the 'powder keg'. a, Multiple giant components, C_1 and C_2, arise simultaneously for the BFW process. The inset shows the number of stable giant components, G, as a function of α, the asymptotic fraction of edges that must be accepted[45]. **b**, The maximum impact from a single edge ΔC_{max} is invariant with system size N for the BFW process, but decays as $N^{-0.065}$ under the product rule process. This holds regardless of whether or not we restrict the BFW process to merge only previously distinct components. **c**, Evolution of the component size distribution, $n(s)$, under the BFW process with $\alpha = 1/2$, showing the build-up of the powder keg, which merges to become two coexisting giant components at t_c.

continuous transitions in the thermodynamic limit, such processes yield significant discrete jumps in the realm of real-world networks.

Modular networks. Several studies show that the paradigm of explosive percolation can be useful for understanding the evolution of modular networks and community structure, including an evolutionary process on the human protein homology network[44]. The general belief is that proteins evolve via duplication–mutation events from ancestral proteins, and it has been shown that more similar (or homologous) proteins organize into network modules[62]. Initializing an evolutionary process on the network with all the proteins disconnected and with edges between the most similar proteins added sequentially leads to the emergence of many large isolated components of tightly connected nodes. These modules eventually link together with the addition of just a few inter-component edges so that global connectivity emerges in an explosive manner. The emergent structure is similar to the dense connectivity within a community and the weak links between communities suggested by Grannoveter for social systems[63].

Monitoring the evolution of an explosive percolation process on a network can also reveal information about the underlying structure, as is the case in a study of empirical data from two real-world social networks—one a mobile-phone call network, the other a co-authorship networks of scientists[64]. Initially, all the empirical edges are considered 'unoccupied' and an Achlioptas process is used to sequentially occupy edges. They show that, at t_c, the component structure reflects the underlying community structure of the network. Thus, applying such graph-evolution processes to data from real-world networks can provide a potential tool for uncovering unknown, underlying structures.

Cluster-driven dynamics are also seen in the first-year growth of many distinct language wikipedias[65]. Each wikipedia is a network of articles connected via hyperlink edges. Most of the languages exhibit the same general pattern of evolving a collection of large disconnected components, with each component focused on a distinct topic. Similar to the 'powder keg', these distinct components quickly link together over the course of a few days, leading to large discrete jumps in the size of the largest component.

Disordered media. Standard formulations of percolation have been used to model many properties of materials and disordered media, such as electrical and thermal conductivity, flow through porous media, and polymerization. Explosive percolation offers a novel ingredient, namely suppressing the growth of the largest components and instead creating many components of uniform size. This allows us to extend percolation models to systems that have not been previously amenable to such treatment.

For example, consider the seminal model of diffusion-limited cluster aggregation[66]. Here clusters move via Brownian motion so that the velocity of a cluster is inversely proportional to the square root of its size, and thus larger clusters move considerably more slowly. Diffusion-limited cluster aggregation can be mapped onto the framework of explosive percolation[67]. In this formulation, clusters move on an underlying two-dimensional lattice via Brownian motion and form larger clusters whenever two clusters become nearest neighbours. Brownian motion suppresses the mobility of the largest clusters, impeding their growth, and leading to the discontinuous emergence of a giant cluster as a function of the number of aggregation events. Generalized Brownian motion, for which the velocity is inversely proportional to the mass of the cluster to a power η, gives rise to a tricritical point that separates discontinuous from continuous emergence as a function of η (ref. 67).

A generalization of the devil's staircase model, called 'fractional percolation', involves systematically suppressing the merging of components with substantially different sizes and preferentially merging those whose size ratio is close to a fixed target ratio f (ref. 42). For any target ratio f (no matter how small) this leads to a series of multiple discontinuous jumps in the supercritical regime. The sizes and locations of the jumps are randomly distributed, similar to crackling noise observed in materials, such as when a sheet of paper is crumpled. This framework links explosive percolation with phenomena that exhibit crackling noise, are non-self-averaging, and exhibit power-law fluctuations resembling Barkhausen noise in ferromagnets.

Recently, the electric breakdown of substrates on which highly conducting particles are adsorbed and desorbed has been identified as a promising candidate for an experimental realization exhibiting a truly discontinuous percolation transition[68].

The behaviours of nanotubes are often modelled by means of standard percolation, where the emergence of percolating paths in bundles of nanotubes captures the transition from insulator to conductor[69]. However, explosive percolation processes are more realistic models, as observations of real-world systems show that the sizes of the bundles are uniform[69]. Similar to explosive percolation processes (and unlike regular percolation), the growth of larger bundles is suppressed and the transition becomes extremely abrupt. The transition shows hysteresis, as is expected for first-order transitions[69].

Recent developments. The cluster-aggregation approach that informs much of the work reviewed here also allows us to study competitive percolation processes on growing networks. Note that in all the percolation models discussed thus far, N is fixed and the graph evolves via edge arrival. In a seminal study appearing in 2001, the impact of node arrival on the Erdős–Rényi process was analysed[70]. Starting from a few seed nodes, a new node arrives at each discrete time step and, with probability $\delta \leq 1$, an edge selected uniformly at random is added to the graph. This leads to an infinite-order percolation transition[70]. Following the same

procedure, but using the 'adjacent edge' Achlioptas process[24] for edge addition, considerably delays the onset of the percolation transition, but retains the smooth, infinite-order transition[71]. Thus, network growth via node arrival allows a significantly delayed percolation transition yet can mitigate the abrupt, explosive nature that typically results from delay interventions[71,72].

Also shown recently is that microscopic patterns in the early evolution of percolation processes can be used to predict the location of the critical point[73]. In particular, the generalized BFW process exhibits peaks in relative variance at well-defined values of t_i (with i an integer), which survive in the thermodynamic limit (see Fig. 4c). The positions of the peaks t_i obey a discrete scale invariance[74] (meaning that scale invariance holds only for a discrete set of magnification factors). We can predict the critical point $t_i \to t_c$ from the discrete scaling relation[73]. Non-self-averaging behaviours can thus provide a powerful predictive tool.

Very recently a strict scaling theory for a wide class of Achlioptas processes was developed using the cluster-aggregation approach, which produces the full set of scaling functions and critical exponents[75]. Even more recently, the necessary conditions that a cluster-merging process must satisfy to produce a discontinuous percolation transition were established, both for transitions of the type shown in Fig. 2b and c (ref. 47). The key ingredient involves whether symmetry is preserved or broken during cluster merging. Finally, we note that novel approaches to traditional percolation based on analysing the matrix describing non-backtracking walks on graphs have recently proved helpful in determining the position of the percolation point, the size of the percolating cluster, and the average cluster size[76–78]. Such approaches may become useful for arbitrarily complex percolation models in the future, including for explosive models.

Future directions. There are many directions for future work on the topic of explosive percolation, ranging from theoretical considerations to more practical aspects of how these processes can help us model, control and understand real-world systems. One direction is how explosive percolation processes can be used for creating and analysing modular networks, furthering the initial studies[44,64]. Ordinary percolation on hierarchical lattices leads to an explosive percolation transition[53] and may also show interesting connections to community structures and clustering phenomena. There is also very limited work concerning explosive percolation on directed networks, with work thus far focused on m-edge Achlioptas processes[79].

A more novel consideration is the range of supercritical properties observed in explosive percolation processes, such as multiple giant components and stochastic staircases. Some mechanisms that yield explosive percolation (growth by overtaking, for example) lead to one phase transition and stable coexisting giant components. Other mechanisms result in unstable coexistence and a family of discontinuous, supercritical transitions. Moreover, the fact that multiple giant components arise in percolation is surprising[80], given the multiplicative coalescence underlying classic processes such as the Erdős–Rényi model. Understanding which mechanisms lead to stable and unstable coexisting giants may provide insight into the evolution of modular networks, such as social networks, and also provide a potential mechanism for controlling gel sizes during polymerization when multiple disconnected polymer gels can be desirable[81]. Other real-world systems that may benefit from, and contribute to, deeper understanding of explosive percolation processes include diffusion-limited cluster aggregation and properties of nanotubes and nanowires. The lack of self-averaging throughout the process and non-convergence in the supercritical regime that is observed for many explosive percolation processes challenges our current notions of percolation. Even for the basic product rule process, there remain many open questions[40,82]. Additional references and challenges for the field of percolation, including explosive percolation, are discussed in other recent reviews[27,83,84].

From a conceptual perspective, the insights gained from explosive percolation processes may help us understand how to better manage and control networks. With our increasing reliance on interdependent systems of networks, from electric power grids, to computer networks, to transportation networks and global financial networks, there is increasing need to understand the systemic risk underlying these engineered networks. Often human operators or regulators intervene with a network's functions or structure in an attempt to delay an undesirable outcome, such as a leak in a dam or a crash in a financial market. Such delay interventions can sometimes be successful, yet at other times lead to unanticipated and disastrous failures. Explosive percolation processes provide a new paradigm for modelling the consequences of repeated, small interventions intended to delay a catastrophe, such as in modern engineered and financial systems[85]. We expect that more exciting behaviours and applications of explosive percolation will continue to be discovered.

Received 11 February 2015; accepted 27 May 2015; published online 1 July 2015

References

1. Stauffer, D. & Aharony, A. *Introduction to Percolation Theory* (Taylor & Francis, 1994).
2. Sahimi, M. *Applications of Percolation Theory* (Taylor & Francis, 1994).
3. Erdős, P. & Rényi, A. On random graphs I. Math. *Debrecen* **6,** 290–297 (1959).
4. Erdős, P. & Rényi, A. On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci.* **5,** 17–61 (1960).
5. Gilbert, E. N. Random graphs. *Ann. Math. Stat.* **30,** 1141–1144 (1959).
6. Ben-Naim, E. & Krapivsky, P. L. Kinetic theory of random graphs: From paths to cycles. *Phys. Rev. E* **71,** 026129 (2005).
7. Bollobás, B. *Random Graphs* 2nd edn (Cambridge Univ. Press, 2001).
8. Azar, Y., Broder, A. Z., Karlin, A. R. & Upfal, E. *Proc. 26th ACM Symp. Theory Comput.* 593–602 (1994).
9. Azar, Y., Broder, A. Z., Karlin, A. R. & Upfal, E. Balanced allocations. *SIAM J. Comput.* **29,** 180–200 (1999).
10. Adler, M., Chakarabarti, S., Mitzenmacher, M. & Rasmussen, L. Parallel randomized load balancing. *Random Struct. Algorithms* **13,** 159–188 (1998).
11. Mitzenmacher, M. The power of two choices in randomized load balancing. *Parallel Distrib. Syst.* **12,** 1094–1104 (2001).
12. Bohman, T. & Frieze, A. Avoiding a giant component. *Random Struct. Algorithms* **19,** 75–85 (2001).
13. von Smoluchowski, M. Drei Vorträge uber Diffusion, Brownsche Molekularbewegung und Koagulation von Kolloidteilchen. *Z. Phys.* **17,** 557–585 (1916).
14. Aldous, D. J. Deterministic and stochastic models for coalescence (aggregation and coagulation): A review of the mean-field theory for probabilists. *Bernoulli* **5,** 3–48 (1999).
15. Spencer, J. & Wormald, N. Birth control for giants. *Combinatorica* **27,** 587–628 (2007).
16. Achlioptas, D., D'Souza, R. M. & Spencer, J. Explosive percolation in random networks. *Science* **323,** 1453–1455 (2009).
17. Newman, M. E. J. & Ziff, R. M. A fast Monte Carlo algorithm for site or bond percolation. *Phys. Rev. E* **64,** 016706 (2001).
18. Ziff, R. M. Explosive growth in biased dynamic percolation on two-dimensional regular lattice networks. *Phys. Rev. Lett.* **103,** 045701 (2009).
19. Cho, Y. S., Kim, J. S., Park, J., Kahng, B. & Kim, D. Percolation transitions in scale-free networks under the Achlioptas process. *Phys. Rev. Lett.* **103,** 135702 (2009).
20. Radicchi, F. & Fortunato, S. Explosive percolation in scale-free networks. *Phys. Rev. Lett.* **103,** 168701 (2009).
21. Ziff, R. M. Scaling behavior of explosive percolation on the square lattice. *Phys. Rev. E* **82,** 051105 (2010).
22. Radicchi, F. & Fortunato, S. Explosive percolation: A numerical analysis. *Phys. Rev. E* **81,** 036110 (2010).
23. Friedman, E. J. & Landsberg, A. S. Construction and analysis of random networks with explosive percolation. *Phys. Rev. Lett.* **103,** 255701 (2009).
24. D' Souza, R. M. & Mitzenmacher, M. Local cluster aggregation models of explosive percolation. *Phys. Rev. Lett.* **104,** 195702 (2010).
25. Cho, Y. S. & Kahng, B. Suppression effect on explosive percolation. *Phys. Rev. Lett.* **107,** 275703 (2011).
26. Riordan, O. & Warnke, L. Achlioptas processes are not always self-averaging. *Phys. Rev. E* **86,** 011129 (2012).

27. Bastas, N., Giazitzidis, P., Maragakis, M. & Kosmidis, K. Explosive percolation: Unusual transitions of a simple model. *Physica A* **407,** 54–65 (2014).
28. Schwarz, J. M., Liu, A. J. & Chayes, L. Q. The onset of jamming as the sudden emergence of an infinite k-core cluster. *Europhys. Lett.* **73,** 560–566 (2006).
29. Toninelli, C., Biroli, G. & Fisher, D. S. Jamming percolation and glass transitions in lattice models. *Phys. Rev. Lett.* **96,** 035702 (2006).
30. Jeng, M. & Schwarz, J. M. Force-balance percolation. *Phys. Rev. E* **81,** 011134 (2010).
31. Cao, L. & Schwarz, J. M. Correlated percolation and tricriticality. *Phys. Rev. E* **86,** 061131 (2012).
32. Nagler, J., Levina, A. & Timme, M. Impact of single links in competitive percolation. *Nature Phys.* **7,** 265–270 (2011).
33. Manna, S. S. & Chatterjee, A. A new route to explosive percolation. *Physica A* **390,** 177–182 (2011).
34. da Costa, R. A., Dorogovtsev, S. N., Goltsev, A. V. & Mendes, J. F. F. Explosive percolation transition is actually continuous. *Phys. Rev. Lett.* **105,** 255701 (2010).
35. Grassberger, P., Christensen, C., Bizhani, G., Son, S-W. & Paczuski, M. Explosive percolation is continuous, but with unusual finite size behavior. *Phys. Rev. Lett.* **106,** 225701 (2011).
36. Lee, H. K., Kim, B. J. & Park, H. Continuity of the explosive percolation transition. *Phys. Rev. E* **84,** 020101 (2011).
37. Tian, L. & Shi, D-N. The nature of explosive percolation phase transition. *Phys. Lett. A* **376,** 286–289 (2012).
38. Riordan, O. & Warnke, L. Explosive percolation is continuous. *Science* **333,** 322–324 (2011).
39. Hooyberghs, H. & Van Schaeybroeck, B. Criterion for explosive percolation transitions on complex networks. *Phys. Rev. E* **83,** 032101 (2011).
40. Riordan, O. & Warnke, L. Achlioptas process phase transitions are continuous. *Ann. Appl. Probab.* **22,** 1450–1464 (2012).
41. Nagler, J., Tiessen, T. & Gutch, H. W. Continuous percolation with discontinuities. *Phys. Rev. X* **2,** 031009 (2012).
42. Schröder, M., Ebrahimnazhad Rahbari, S. H. & Nagler, J. Crackling noise in fractional percolation. *Nature Commun.* **4,** 2222 (2013).
43. Cho, Y. S., Hwang, S., Herrmann, H. J. & Kahng, B. Avoiding a spanning cluster in percolation models. *Science* **339,** 1185–1187 (2013).
44. Rozenfeld, H. D., Gallos, L. K. & Makse, H. A. Explosive percolation in the human protein homology network. *Eur. Phys. J. B* **75,** 305–310 (2010).
45. Chen, W. & D'Souza, R. M. Explosive percolation with multiple giant components. *Phys. Rev. Lett.* **106,** 115701 (2011).
46. Araújo, N. A. M. & Herrmann, H. J. Explosive percolation via control of the largest cluster. *Phys. Rev. Lett.* **105,** 035701 (2010).
47. Cho, Y. S. & Kahng, B. Types of discontinuous percolation transitions in cluster merging processes. *Sci. Rep.* http://dx.doi.org/10.1038/srep11905 (in the press).
48. Bohman, T., Frieze, A. & Wormald, N. C. Avoidance of a giant component in half the edge set of a random graph. *Random Struct. Algorithms* **25,** 432–449 (2004).
49. Chen, W. *et al.* Phase transitions in supercritical Explosive Percolation. *Phys. Rev. E* **87,** 052130 (2013).
50. Chen, W. *et al.* Unstable supercritical discontinuous percolation transitions. *Phys. Rev. E* **88,** 042152 (2013).
51. Chen, W., Zheng, Z. & D'Souza, R. M. Deriving an underlying mechanism for discontinuous percolation. *Europhys. Lett.* **100,** 66006 (2012).
52. Panagiotou, K., Spöhel, R., Steger, A. & Thomas, H. Explosive percolation in Erdős-Rényi-like random graph processes. *Electron. Notes Discrete Math.* **38,** 699–704 (2011).
53. Boettcher, S., Singh, V. & Ziff, R. M. Ordinary percolation with discontinuous transitions. *Nature Commun.* **3,** 787 (2012).
54. Moreira, A. A., Oliveira, E. A., Reis, S. D. S., Herrmann, H. J. & Andrade, J. S. Hamiltonian approach for explosive percolation. *Phys. Rev. E* **81,** 040101(R) (2010).
55. Buldyrev, S. V., Parshani, R., Paul, G., Stanley, H. E. & Havlin, S. Catastrophic cascade of failures in interdependent networks. *Nature* **464,** 1025–1028 (2010).
56. Angst, S., Dahmen, S. R., Hinrichsen, H., Hucht, A. & Magiera, M. P. Explosive ising. *J. Stat. Mech.* **2012,** L06002 (2012).
57. Gómez-Gardeñes, J., Gómez, S., Arenas, A. & Moreno, Y. Explosive synchronization transitions in scale-free networks. *Phys. Rev. Lett.* **106,** 128701 (2011).
58. Guan, S., Zhang, X., Boccaletti, S. & Liu, Z. Explosive synchronization in adaptive and multilayer networks. *Phys. Rev. Lett.* **114,** 038701 (2015).
59. Bizhani, G., Paczuski, M. & Grassberger, P. Discontinuous percolation transitions in epidemic processes, surface depinning in random media, and Hamiltonian random graphs. *Phys. Rev. E* **86,** 011128 (2012).
60. Krapivsky, P. L., Redner, S. & Ben-Naim, E. *A Kinetic View of Statistical Physics* (Cambridge Univ. Press, 2010).
61. Cho, Y. S., Kahng, B. & Kim, D. Cluster aggregation model for discontinuous percolation transition. *Phys. Rev. E* **81,** 030103(R) (2010).
62. Medini, D., Covacci, A. & Donati, C. Protein homology network families reveal step-wise diversification of Type III and Type IV secretion systems. *PLoS Comput. Biol.* **2,** e173 (2006).
63. Granovetter, M. S. The strength of weak ties. *Am. J. Soc.* 1360–1380 (1973).
64. Pan, R. K., Kivelä, M., Saramäki, J., Kaski, K. & Kertész, J. Using explosive percolation in analysis of real-world networks. *Phys. Rev. E* **83,** 046112 (2011).
65. Bounova, G. A. *Topological Evolution of Networks: Case Studies in the US Airlines and Language Wikipedias* PhD thesis, Massachusetts Institute of Technology (2009).
66. Witten, T. A. Jr & Sander, L. M. Diffusion-limited aggregation, a kinetic critical phenomenon. *Phys. Rev. Lett.* **47,** 1400–1403 (1981).
67. Cho, Y. S. & Kahng, B. Discontinuous percolation transitions in real physical systems. *Phys. Rev. E* **84,** 050102 (2011).
68. Oliveira, C. L. N., Araújo, N. A., Andrade, J. S. & Herrmann, H. J. Explosive electric breakdown due to conducting-particle deposition on an insulating substrate. *Phys. Rev. Lett.* **113,** 155701 (2014).
69. Kim, Y., Yun, Y-k. & Yook, S-H. Explosive percolation in a nanotube-based system. *Phys. Rev. E* **82,** 061105 (2010).
70. Callaway, D. S., Hopcroft, J. E., Kleinberg, J. M., Newman, M. E. J. & Strogatz, S. H. Are randomly grown graphs really random? *Phys. Rev. E* **64,** 041902 (2001).
71. Vijayaraghavan, V. S., Noël, P-A., Waagen, A. & D'Souza, R. M. Growth dominates choice in network percolation. *Phys. Rev. E* **88,** 032141 (2013).
72. Do Yi, S., Jo, W. S., Kim, B. J. & Son, S-W. Percolation properties of growing networks under an Achlioptas process. *Europhys. Lett.* **103,** 26004 (2013).
73. Chen, W., Schröder, M., D'Souza, R. M., Sornette, D. & Nagler, J. Microtransition cascades to percolation. *Phys. Rev. Lett.* **112,** 155701 (2014).
74. Sornette, D. Discrete scale invariance and complex dimensions. *Phys. Rep.* **297,** 239–270 (1998).
75. da Costa, R. A., Dorogovtsev, S. N., Goltsev, A. V. & Mendes, J. F. F. Solution of the explosive percolation quest: Scaling functions and critical exponents. *Phys. Rev. E* **90,** 022145 (2014).
76. Hamilton, K. E. & Pryadko, L. P. Tight lower bound for percolation threshold on an infinite graph. *Phys. Rev. Lett.* **113,** 208701 (2014).
77. Karrer, B., Newman, M. E. J. & Zdeborová, L. Percolation on sparse networks. *Phys. Rev. Lett.* **113,** 208702 (2014).
78. Radicchi, F. Predicting percolation thresholds in networks. *Phys. Rev. E* **91,** 010801 (2015).
79. Squires, S. *et al.* Weakly explosive percolation in directed networks. *Phys. Rev. E* **87,** 052127 (2013).
80. Spencer, J. The giant component: The golden anniversary. *Not. AMS* **57,** 720–724 (2010).
81. Ben-Naim, E. & Krapivsky, P. L. Percolation with multiple giant clusters. *J. Phys. A* **38,** L417 (2005).
82. Riordan, O. & Warnke, L. The evolution of subcritical Achlioptas processes. *Random Struct. Algorithms* http://dx.doi.org/10.1002/rsa.20530 (2014).
83. Araújo, N. A. M., Grassberger, P., Kahng, B., Schrenk, K. J. & Ziff, R. M. Recent advances and open challenges in percolation. *Eur. Phys. J. Spec. Top.* **223,** 2307–2321 (2014).
84. Saberi, A. A. Recent advances in percolation theory and its applications. *Phys. Rep.* **578,** 1–32 (2015).
85. Helbing, D. Globally networked risks and how to respond. *Nature* **497,** 51–59 (2013).

Acknowledgements

We thank M. Schröder and A. Witt for valuable discussions and assistance in the preparation of the figures, and C. D'Souza and L. Nagler-Deutsch for invaluable input. R.M.D'S. gratefully acknowledges support from the US Army Research Office MURI Award No. W911NF-13-1-0340 and Cooperative Agreement No. W911NF-09-2-0053 and the Defense Threat Reduction Agency Basic Research Award HDTRA1-10-1-0088.

Additional information

Reprints and permissions information is available online at www.nature.com/reprints. Correspondence should be addressed to R.M.D'S. or J.N.

Competing financial interests

The authors declare no competing financial interests.