Problem Set 2 Solutions

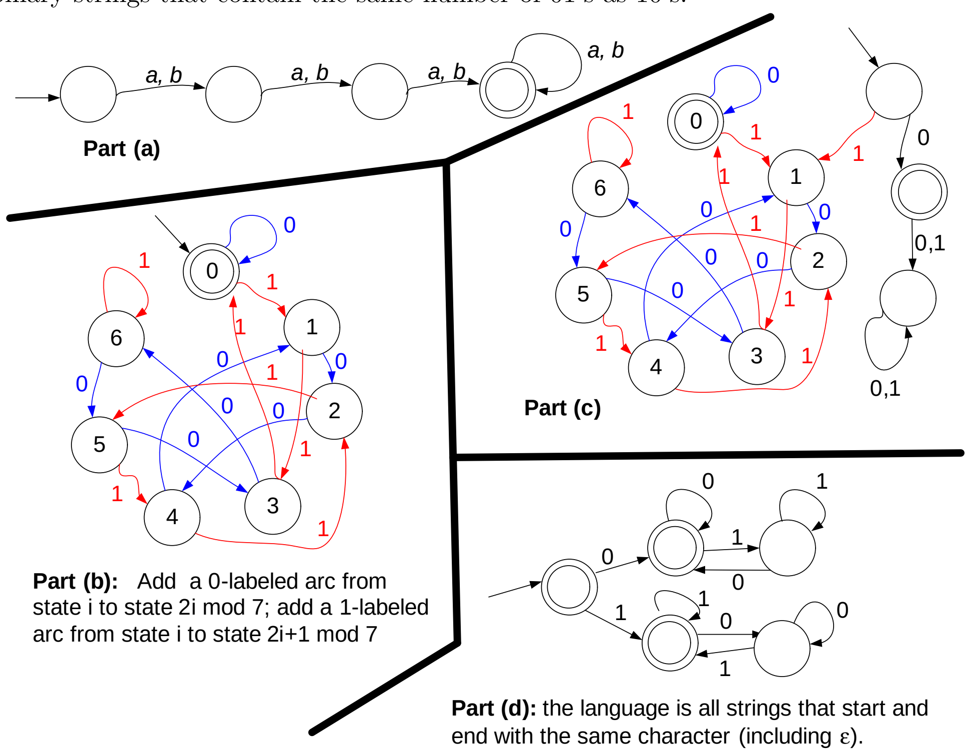
Problem 1 *Draw DFAs for the following languages:*

(a) $A = \{x \in \{a,b\}^* : \ |x| \geq 3\}$

(b) B = the binary encodings of numbers divisible by 7. Allow leading zeros and the empty string as alternate names of 0. Thus $B = \{\varepsilon, 0, 00, 000, 111, 0000, 1110, \ldots\}$

(c) C = the binary encodings of numbers divisible by 7. Don't allow leading zeros or the empty string. Thus $C = \{0, 111, 1110, \ldots\}$.

(d) D = binary strings that contain the same number of 01's as 10's.



Problem 2. *Let $\mathcal{E}(L) = \{x \in L :$ there exists a $y \in \Sigma^+$ for which $xy \in L\}$. (By Σ^+ we mean $\Sigma\Sigma^*$.)*

Part A. *What is $\mathcal{E}(\{0,1\}^*)$? What is $\mathcal{E}(\{\varepsilon, 0, 1, 00, 01, 111, 1110, 1111\})$?*

$\mathcal{E}(\{0,1\}^*) = \{0,1\}^*$, while $\mathcal{E}(\{\varepsilon, 0, 1, 00, 01, 111, 1110, 1111\}) = \{\varepsilon, 0, 1, 111\}$.

Part B. *Prove that if L is DFA-acceptable then $\mathcal{E}(L)$ is, too.*

Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$ for L, a DFA $M = (Q, \Sigma, \delta, q_0, F')$ is constructed for $\mathcal{E}(L)$ by "pruning" the final state set; we define F' to be the set of all states $q \in F$ such that there exists some nontrivial path from q to some final state of M. Then $x \in L(M')$ iff $x \in L$ and there is some $y \in \Sigma^+$ such that $xy \in L(M)$.

Problem 3 *State whether the following propositions are true or false, proving each answer.*

(a) *Every DFA-acceptable language can be accepted by a DFA with an odd number of states.*

True. The idea is to add a "dummy state" in the case that the machine has an even number of states. Formally, given a DFA $M = (Q, \Sigma, \delta, q_0, F)$, if $|Q|$ is odd set $M' = M$ and if $|Q|$ is even then let $M' = (Q \cup \{s\}, \Sigma, \delta', q_0, F)$ (where $s \notin Q$) and let $\delta'(q, a) = \delta(q, a)$ for $q \in Q, a \in \Sigma$ and $\delta'(s, a) = s$ (say) for $a \in \Sigma$. Clearly $L(M) = L(M')$ and M' has an odd number of states.

(b) *Every DFA-acceptable language can be accepted by a DFA whose start state is never visited twice.*

True. Add a new start state and connect it up to all the states that the old start state was connected to, in the same way. Formally, given a DFA $M = (Q, \Sigma, \delta, q_0, F)$ construct a DFA $M' = (Q \cup \{s\}, \Sigma, \delta', s, F')$ (for $s \notin Q$) by saying $\delta'(q, a) = \delta(q, a)$ for $q \in Q$, $a \in \Sigma$, and $\delta'(s, a) = \delta(q_0, a)$ for $a \in \Sigma$, and $F' = F$ if $q_0 \notin F$ and $F = F \cup \{s\}$ if $q_0 \in F$.

(c) *Every DFA-acceptable language can be accepted by a DFA no state of which is ever visited more than once.*

False. Only finite languages can be accepted by such a machine, and some DFA-acceptable languages are infinite.

(d) *The language $L = \{x \in \{a, b\}^* : x$ starts and ends with the same character$\}$ can be accepted by a DFA $M = (Q, \Sigma, \delta, q_0, F)$ for which $\delta^*(q_0, w) = q_0$ for some $w \neq \varepsilon$. Assume an alphabet of $\Sigma = \{a, b\}$.*

False. Since $a \in L$ and $b \in L$ we know that $\delta(q_0, a) \in F$ and $\delta(q_0, b) \in F$. If w begins with an a then $\delta^*(q_0, wb) = \delta^*(q_0, b) \in F$, but $wb \notin L$. If w begins with a b then $\delta^*(q_0, wa) = \delta^*(q_0, a) \in F$, but $wa \notin L$.

Problem 4 *A homomorphism is a function $h : \Sigma \to \Gamma^*$ for alphabets Σ, Γ. Given a homomorphism h, extend it to strings and then languages by asserting that $h(\varepsilon) = \varepsilon$, $h(a_1 \cdots a_n) = h(a_1) \cdots h(a_n)$ (for $a_1, \ldots, a_n \in \Sigma$), and $h(L) = \{h(x) : x \in L\}$.*

(a) *Prove: for any homomorphism h, if L is DFA-acceptable, then so is $h(L)$.*

Given a DFA M, replace each a-labeled transition by an $h(a)$-labeled one. For arcs now bearing multi-character labels $a_1 \cdots a_m$, add $m-1$ intermediate states connected by arcs labeled by $a_1, \ldots, a_m$. In this way we get an NFA for $h(L)$. The equivalence of the DFA and NFA acceptable languages establishes the result.

(b) *Disprove: for any homomorphism h, if $h(L)$ is DFA-acceptable, then so is L. For this you may assume that there's a language L that is not DFA-acceptable.*

Let $L \subseteq \Sigma^*$ be any language that is *not* DFA-acceptable. Let $h(a) = \varepsilon$ for all $a \in \Sigma$. Then $h(L) = \{\varepsilon\}$ is DFA-acceptable even though L is not.

Problem 5. *Fix a DFA $M = (Q, \Sigma, \delta, q_0, F)$. For any two states $q, q' \in Q$, let us say that q and q' are* equivalent, *written $q \sim q'$, if, for all $w \in \Sigma^*$ we have that $\delta^*(q, w) \in F \Leftrightarrow \delta^*(q', w) \in F$. Here δ^* is the extension of δ to Σ^* defined by $\delta^*(q, \varepsilon) = q$ and $\delta^*(q, ax) = \delta^*(\delta(q, a), x)$.*

(a) *Prove that $\sim$ is an equivalence relation.*

This is immediate from the definition. Reflexive: $q \sim q$ because $\delta^*(q, w) \in F$ iff $\delta^*(q, w) \in F$. Symmetric: If $q \sim q'$ then, for all $w \in \Sigma^*$, $\delta^*(q, w) \in F$ iff $\delta^*(q', w) \in F$. Thus $q' \sim q$. Transitive: If $q \sim q'$ and $q' \sim q''$ then, for all $w \in \Sigma^*$, $\delta^*(q, w) \in F$ iff $\delta^*(q', w) \in F$ iff $\delta^*(q'', w) \in F$.

(b) *Suppose that $q \sim q'$ for distinct q, q'. Describe, first in plain English and then in precise mathematical terms, how to construct a smaller (=fewer state) DFA M' that accepts the same language as M.*

Create M' by eliminating q' and redirecting all arcs into it into state q, instead. Formally, assuming $q' \neq q_0$, let $M' = (Q', \Sigma, \delta', q_0, F')$ where $Q' = Q - \{q'\}$, $F' = F - \{q'\}$, and $\delta'(p, a) = q$ when $\delta(p, a) = q'$, and $\delta'(p, a) = \delta(p, a)$ otherwise. If $q' = q_0$ then swap the roles of q and q' (or change the start state to q).