Quiz 9 Solutions

For this quiz I want you to prove that

\[ A = \{ \langle M, k \rangle : M \text{ is a TM that accepts at least one string of length } k \} \]

is undecidable. Do this with a reduction involving \( A_{\text{TM}} \) or \( \overline{A_{\text{TM}}} \). Make your proof succinct, legible, and logical. Write exclusively in grammatical English sentences.

Setup. Since \( A \) is r.e., we will show that it is undecidable by showing that \( A_{\text{TM}} \leq_m A \). To do this, we must construct a Turing-computable function that maps a string \( \langle M, w \rangle \) to a string \( \langle M', k \rangle \) such that TM \( M \) accepts \( w \) if and only if TM \( M' \) accepts some string of length \( k \).

Construction. Given \( \langle M, w \rangle \) the reduction returns \( \langle M', k \rangle \) where \( k \geq 0 \) is an arbitrary fixed value, say \( k = 0 \), and TM \( M' \) is the following machine:

Machine \( M' \), on input \( x \):
- Run \( M \) on \( w \)
- If \( M \) accepts then accept
- If \( M \) rejects then reject

Analysis. If \( M \) accepts \( w \) then we will have that \( L(M') = \Sigma^* \), so \( M' \) will accept a string of length \( k \) (as it accepts all strings of all lengths). On the other hand, if \( M \) does not accept \( w \) then \( L(M') = \emptyset \) so \( M' \) will not accept any string of length \( k \) (as it accepts no string of any length). Finally, the function that computes \( \langle M', k \rangle \) from \( \langle M, w \rangle \) is clearly computable.