Quiz 2 tomorrow & PS4

- Q2: review PS1-3 and quiz 1, especially difficult questions, definitions from class
- PS4 Q5: calculus may be used; Q8c $\subsetneq$ **proper subset** of, $A \subsetneq B$ means that A is a subset of but not equal to B

Mathematical Induction

To prove proposition $P(n)$ for all integers $n \geq n_0$, prove the basis: $n = n_0$ is true
Inductive step: suppose $n = k$ is true, show $n = k + 1$ is true

1. Prove that $n^3 + 5n$ is divisible by 6 for every $n \geq 1$.

   Basis: $n = 1$: $6 \mid 1^3 + 5(1)$
   Inductive step: suppose $n = k$ is true, so $6 \mid k^3 + 5k$
   Check $n = k + 1$:
   $$\begin{aligned}(k+1)^3 + 5(k+1) &= (k^3 + 3k^2 + 3k + 1) + (5k + 5)\\ &= (k^3 + 5k) + (3k^2 + 3k + 6)\\ &= (k^3 + 5k) + 3k(k+1) + 6\end{aligned}$$
   From the inductive hypothesis, $6 \mid (k^3 + 5k)$. $k(k+1)$ is even, so $6 \mid 3k(k+1)$, and $6 \mid 6$. Inductive step verified.

2. Prove that $8^n - 3^n$ is divisible by 5 for every $n \geq 1$.

   Basis: $n = 1$: $5 \mid 8^1 - 3^1$
   Inductive step: suppose $n = k$ is true, so $5 \mid 8^k - 3^k$
   Check $n = k + 1$:
   $$\begin{aligned}8^{k+1} - 3^{k+1} &= 8 * 8^k - 3 * 3^k\\ &= 8 * 8^k - 3 * 8^k + 3 * 8^k - 3 * 3^k\\ &= (8 - 3) * 8^k + 3 * \left(8^k - 3^k\right)\\ &= 5 * 8^k + 3 * (8^k - 3^k)\end{aligned}$$
   We know that $5 \mid 5 * 8^k$. From the inductive hypothesis, $5 \mid (8^k - 3^k)$. Inductive step verified.

3. Prove that $n^2 > 7n + 1$ for all $n \geq 8$.

   Basis: $n = 8$: $64 > 56 + 1$
   Inductive step: suppose $n = k$ is true, so $k^2 > 7k + 1$
   Check $n = k + 1$, $(k+1)^2 > 7(k+1) + 1$
   LHS: $(k+1)^2 = k^2 + 2k + 1 > (7k + 1) + 2k + 1 = 7(k+1) + 1 + (2k - 6)$
   Since $k \geq 8$, $2k - 6 > 0$ so LHS is greater than RHS, inductive step verified.

4. **Induction on a definition** An expression E is defined as
   $E \rightarrow < Number >$
   $E \rightarrow (E + E) \mid (E * E) \mid (E \hat{} E) \mid \ln(E)$

Prove that an expression E has the same number of (and).
<u>Basis</u>: $E = < Number >$, 0 ('s and)'s
<u>Inductive step</u>: suppose $n = k$ is true, E_k has the same number of (and).
Check $n = k + 1$, E_{k+1} :
Take on either the form $(E + E)$ | $(E * E)$ | $(E^\wedge E)$ | $\ln(E)$, all of which contain an equal number of (and). Inductive step verified.

<u>Strong vs. Weak Induction</u>

- method so far is sometimes called **weak induction**, $P(k) \rightarrow P(k+1)$
- **strong induction** shows $(\forall n)P(n)$ by showing $P(1) \wedge \ldots \wedge P(k) \rightarrow P(k+1)$
- both approaches are equally "strong", strong induction is easier for certain proofs, such as PS4 Q6

5. Prove that $a_n \leq 2^n$ for the sequence $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, $a_i = a_i + a_{i-1} + a_{i-2}$
<u>Basis</u>: $n = 0,1,2$ true because $1 \leq 2^0$, $2 \leq 2^1$, $3 \leq 2^2$
<u>Inductive step</u>: assume $n = 0 \ldots (k-1)$ is true, show that $n = k$ is true
i.e. show that $P(k) = a_k \leq 2^k$

$$\begin{aligned} a_k &= a_{k-1} + a_{k-2} + a_{k-3} \\ &\leq 2^{k-1} + 2^{k-2} + 2^{k-3} \\ &\leq 2^0 + 2^1 + \cdots + 2^{k-3} + 2^{k-2} + 2^{k-1} \end{aligned}$$

Geometric series: $\leq 2^k - 1 \leq 2^k$

<u>Envelope substitution question</u>

A store sells envelopes in packages of 5 and 12. Prove that for $\geq n$, the store can sell exactly n envelopes. To find n without trying every number, start with the smallest substitutions to make one more envelope (=inductive step).

The lowest substitution going from 5- to 12-packs is $7(5) \rightarrow 3(12)$. This substitution needs at least 7 5-packs.

Consider the case with less than 7 5-packs, i.e. at most 6 5-packs (≤ 30). The lowest substitution going from 12- to 5-packs is $2(12) \rightarrow 5(5)$. This substitution needs at least 2 12-packs (≥ 24).

To find n, we need only to check $30 < n \leq 54$, since there will be numbers below 30 that cannot be achieved, and any number 54 or above will work for certain. We will find $n = 44$ to be the first number beyond which all n is satisfied.

We can also reverse the order of substitutions. In this example that would be checking $12 < n \leq 47$, which also includes $n = 44$.

Once n is found, we can formally prove our hypothesis by induction using n as the basis.