Solving recurrence relations with repeated substitutions

1. Solve $T(n) = 2T\left(\frac{n}{3}\right) + n$ to within a $\Theta(\cdot)$ result.

Substituting in $\frac{n}{3}$ for n in the equation, we have $T\left(\frac{n}{3}\right) = 2T\left(\frac{n}{3^2}\right) + \frac{n}{3}$

From the original equation:

$$T(n) = 2T\left(\frac{n}{3}\right) + n$$

$$T(n) = 2\left[2T\left(\frac{n}{3^2}\right) + \frac{n}{3}\right] + n$$

$$T(n) = 2^2T\left(\frac{n}{3^2}\right) + n\left(1 + \frac{2}{3}\right)$$

Substituting in $\frac{n}{3^2}$ for n in the equation, we have $T\left(\frac{n}{3^2}\right) = 2T\left(\frac{n}{3^3}\right) + \frac{n}{3^2}$

$$T(n) = 2^2\left[2T\left(\frac{n}{3^3}\right) + \frac{n}{3^2}\right] + n\left(1 + \frac{2}{3}\right)$$

$$T(n) = 2^3T\left(\frac{n}{3^3}\right) + n\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2\right)$$

$$T(n) = 2^kT\left(\frac{n}{3^k}\right) + n\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^{k-1}\right)$$

The series $1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots$ is a geometric series (where $r < 1$). $s = \frac{1}{1-r} = \frac{1}{1-\frac{2}{3}} = 3$. So right term is $\Theta(\mathrm{n})$.

For left term, use $k = \log_3 n$:

$$2^{\log_3 n}T\left(\frac{n}{3^{\log_3 n}}\right) = 2^{\log_3 n}T(1)$$

For a small enough n, $T(n)$ has constant runtime $\Theta(1)$, so we can ignore $T(1)$.

$$2^{\log_3 n} = \left(3^{\log_3 2}\right)^{\log_3 n} = \left(3^{\log_3 n}\right)^{\log_3 2} = n^{\log_3 2}$$

Comparing the two terms, $\Theta(n^{\log_3 2}) < \Theta(\mathrm{n})$, therefore $\boldsymbol{T(n) = \Theta(n)}$.

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2. Solve $T(n) = 3T\left(\frac{n}{2}\right) + n$ to within a $\Theta(\cdot)$ result.

$$T(n) = 3\left[3T\left(\frac{n}{2^2}\right) + \frac{n}{2}\right] + n$$

$$T(n) = 3^2T\left(\frac{n}{2^2}\right) + n\left(1 + \frac{3}{2}\right)$$

$$T(n) = 3^3T\left(\frac{n}{2^3}\right) + n\left(1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2\right)$$

$$T(n) = 3^kT\left(\frac{n}{2^k}\right) + n\left(1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \cdots + \left(\frac{3}{2}\right)^{k-1}\right)$$

Now consider the series $S = 1 + x + x^2 + \cdots + x^{k-1} + x^k$:

$$Sx = x + x^2 + \cdots + x^{k-1} + x^k + x^{k+1}$$

$$1 + Sx = 1 + x + x^2 + \cdots + x^{k-1} + x^k + x^{k+1}$$

$$1 + Sx = S + x^{k+1}$$

$$Sx - S = x^{k+1} - 1$$

$$S(x-1) = x^{k+1} - 1$$

$$S = \frac{x^{k+1} - 1}{x - 1}$$

$$n\left(1+\frac{3}{2}+\left(\frac{3}{2}\right)^2+\cdots+\left(\frac{3}{2}\right)^{k-1}\right)=\frac{\frac{3}{2}^k-1}{\frac{1}{2}}=2\left(\left(\frac{3}{2}\right)^k-1\right)$$

$$T(n)=3^kT\left(\frac{n}{2^k}\right)+2n\left(\left(\frac{3}{2}\right)^k-1\right)$$

Use $k=\lg n$:

$$T(n)=3^{\lg n}T\left(\frac{n}{2^{\lg n}}\right)+2n\left(\left(\frac{3}{2}\right)^{\lg n}-1\right)$$

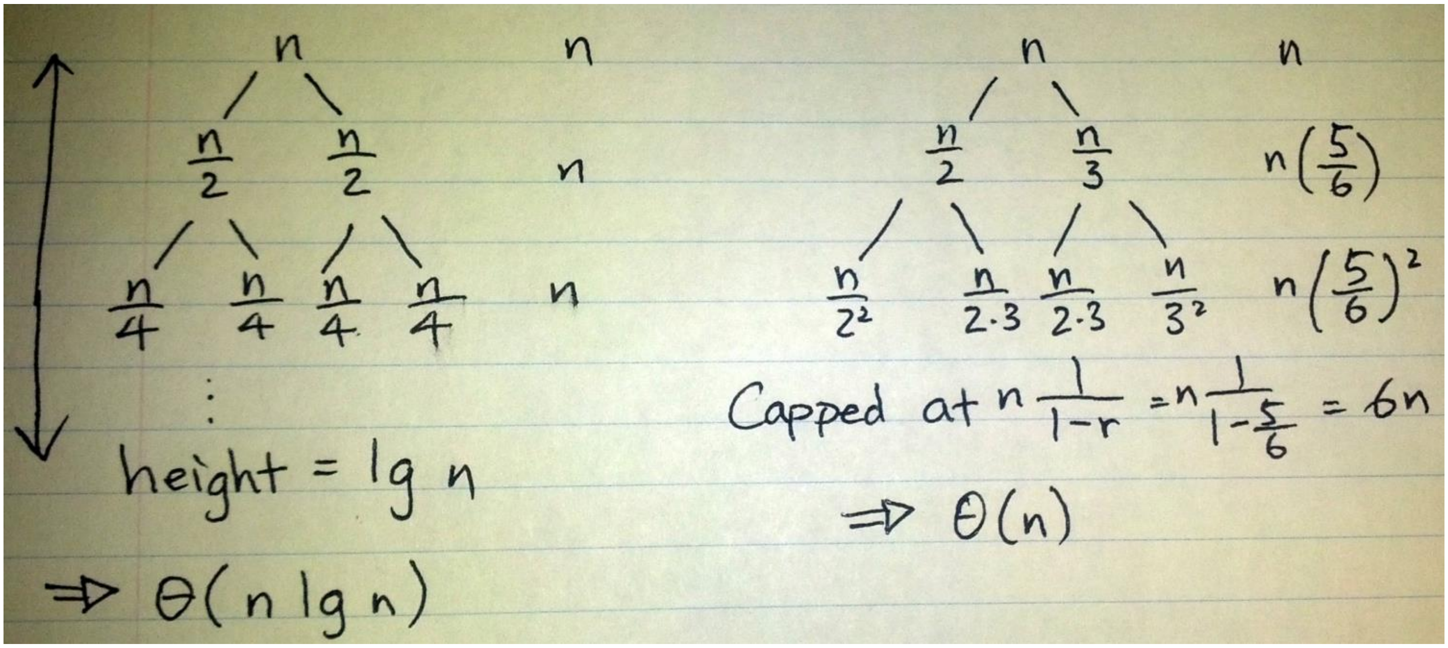
$$\Theta\big(T(n)\big)=\Theta(3^{\lg n})+\Theta\left(2n\left(\frac{3}{2}\right)^{\lg n}-1\right)$$

$$\Theta\big(T(n)\big)=\Theta\big((2^{\lg 3})^{\lg n}\big)+\Theta\left(2n\left(\frac{3^{\lg n}}{2^{\lg n}}\right)-2n\right)$$

$$\Theta\big(T(n)\big)=\Theta\big(n^{\lg 3}\big)+\Theta\big(2*3^{\lg n}-2n\big)=\Theta\big(n^{\lg 3}\big)+\Theta\big(n^{\lg 3}\big)=\boldsymbol{\Theta\big(n^{\lg 3}\big)}$$

Solving recurrence relations with recursion trees

Solve $T(n)=2T\left(\frac{n}{2}\right)+n$ to within a $\Theta(\cdot)$ result. Solve $T(n)=T\left(\frac{n}{2}\right)+T\left(\frac{n}{3}\right)+n$ to within a $\Theta(\cdot)$ result.



To determine the height, consider the number of recursions to get to $T(1)$, at which point the recursion stops.

For $T(n)=2T\left(\frac{n}{2}\right)+n$, the height is $\lg n$ since n is being divided by two each time. There are $\lg n$ levels each taking n time, so it is $\Theta(n\lg n)$.

For $T(n)=T\left(\frac{n}{2}\right)+T\left(\frac{n}{3}\right)+n$ we do not need to consider the height, because we see that the total runtime is n times a constant number. Its runtime is capped at 6n, so it is $\Theta(n)$.

Big-O and Theta

$O(\cdot)$ is an upper bound. $\Theta(\cdot)$ is both an upper and lower bound (or tight bound). True or false?

Note: $>$ is not defined. $\Theta(f) > \Theta(g)$ has been re-written to $g \in O(f)$.

a) $n = O(n)$ **T**
b) $n = \Theta(n)$ **T**
c) $n = O(n^2)$ **T**
d) $n = \Theta(n^2)$ **F**
e) $n^2 = O(n)$ **F**
f) $n^2 = \Theta(n)$ **F**
g) $100\text{n} \in O(0.01n^2)$ **T**
h) $2^{\text{n}} \in O\left(2^{\sqrt{n}}\right) >$ **F**
i) $\text{n}^{\lg n} \in O(2^n)$ **T**
j) $n! \in O(2^n)$ **F**
k) $\ln n \in O(\lg n)$ **T**
l) $\log n \in O(\ln n)$ **T**

For h), $\Theta\left(2^{\sqrt{n}}\right)$ and $\Theta(2^n)$ do have two different growth rates.

For k) and l), since $\log_b x = \log_d x \,/\, \log_d b$, different logarithmic bases only differ by a multiplicative constant. So $\Theta(\lg n) = \Theta(\ln n) = \Theta(\log n)$. Alternatively $\lg n, \ln n, \log n \in \mathrm{O}(\lg n)$.

Rank the following functions by order of growth: 1, n, n^2, n^3, $\lg n$, $\ln n$, $\lg\lg n$, $\ln\ln n$, 2^n, $n\lg n$, $n^{\lg n}$

1
$\ln\ln n \quad \lg\lg n$
$\ln n \quad \lg n$
n
$n\lg n$
n^2
n^3
$n^{\lg n}$
2^n