ECS 20 — Lecture 19 — Fall 2013 —3 Dec 2013

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Today:

- Probability

Announcements:

- Please work out the old final for Thursday
- Final's week OH are online. Unfortunately, I am out of town. I will check the chat room as often as I can.
- Final: you may not leave during the last 30 mins.

Poker examples and first use of probability

Let's introduce probability informally

straight flush = five **consecutive** cards: A2345, 23456, 34567, ..., 89AJQ, 9AJQK, AJQKA in any suit. *So 40 possible*.
royal flush = AJQKA of one suit. So *4 possible*
four of a kind = four cards of one value, eg., four 9's
full house = 3 cards of one value, 2 cards of another value. Eg, three 10's and two 4's.
flush = five cards of a single suit
three of a kind = three cards of one value, a fourth card of a different value, and a fifth card of a third value
two pairs = two cards of one value, two more cards of a second value, and the remaining card of a third value
one pair = two cards of one value, but not classified above

1. **How many poker hands** are there?

   Answer: $C(52,5)$=2,598,960

2. How many poker hands are **full houses**?
   Answer: A full house can be **partially** identified by a pair, like (J,8), where the first component of the pair is what you have **three** of, the second component is what you have **two** of. So there are $P(13,2)$=13*12 such pairs. For each there are $C(4,3)$=4 ways to choose the first component, and $C(4,2)$=6 ways to choose the second component. So all together there are

   13*12*4*6=3,744 possible full houses.

   The probability of being dealt a full house is therefore
   3,744/2,598,960 ≈ .001441 ≈ 0.14 %

   P[FullHouse] ≈ .001441

The probability of an event is a real number between 0 and 1 (inclusive). If asked what's the probability of something, don't answer with a "percent", and don't answer with something outside of [0,1]. When we give something in "percent's", we are giving a probability multiplied by 100.

3. How many poker hands are **two pairs**?

   Answer: We can partially identify two pairs as in {J, 8}. Note that now the pair is now **unordered**. There are $C(13,2)$ such sets. For each there are $C(4,2)$ ways to choose the larger card and $C(4,2)$ ways to choose the smaller card. There are now 52 – 8 remaining cards one can choose as the fifth card (to avoid a full house, there are 8 "forbidden" cards). So the total is

   $$C(13,2)*C(4,2)*C(4,2)*44 = 123{,}552.$$

   The chance of being dealt two pairs is therefore
   $C(13,2)*C(4,2)*C(4,2)*44 \,/\, C(52,5) = 123{,}552/2{,}598{,}960 \approx 0.047539 \approx 4.75\%$

   $P[\text{TwoPairs}] \approx 0.047539$

Basic definitions / theory

Schaum's, chapter 7.
Probability does not appear at all in Biggs.

Def: A (finite) **probability space** (S,P) is

- a finite set S (*the sample space*) and
- a function P: $S \rightarrow [0,1]$ (the *probability measure*) such that that

$$\sum_{x \in S} P(x) = 1$$ (alternative notation: ω, μ, Ω for x, P, S)

In general, whenever you hear *probability* make sure that you are clear **what** is the probability space is: what is the sample space and what is the probability measure on it.
An **outcome** is a point in S.

Def: Let (S, P) be a probability space.
An **event** is a subset of S.

Def: Let A be an event of probability space (S, P).

$$P(A) = \sum_{a \in A} P(a)$$ (I'm used to using Pr, will probability slip)

The probability of event A. By convention, $P(\emptyset)=0$

Def: The **uniform** distribution is the one where $P(a) = 1/|S|$ —ie, all points are equiprobable.
Def: Events A and B are **independent** if $P(A \cap B) = P(A)\,P(B)$.

Def: A **random variable** is a function $X\colon S \to \mathbf{R}$ from the sample space to the reals.

Def: $\mathbf{E}[X] = \sum_{s \in S} P(s)\, X(s)$ // expected value of X ("average value")

Def: if $B \neq \varnothing$ then $P(A|B) = P(A \cap B) / P(B)$

Propositions:

- $P(\varnothing) = 0$ // by definition
- $P(S) = 1$
- $P(A) + P(A^c) = 1$, or $P(A) = 1 - P(A^c)$

- If A and B are disjoint events (that is, disjoint sets) then $P(A \cup B) = P(A) + P(B)$
- ("**sum bound**")
  $P(A \cup B) \leq P(A) + P(B)$
- In general,
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ // inclusion-exclusion principle

- If $B1$,$B2$ disjoint, nonempty events whose union is S then
  $P(A) = P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2)$

- $\mathbf{E}(X+Y) = \mathbf{E}(X) + \mathbf{E}(Y)$ // expectation is linear.

Eg 1: Dice

The singular, the students assure me, is *die*.
Like *mice* and *mie*. I guess.

- You roll a fair die six times:
  $S = \{1,2,3,4,5,6\}$
  $P(1)=P(2)=...=P(6)=1/6$

  "you roll an even number" is an event.
  Event is $A = \{2,4,6\}$. $P(A) = 3 * (1/6) = 1/2$.

- Pair of dice.
  $S = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$o
  $P((a,b)) = 1/36$ for all (a,b) in S

Illustrate independence.

P(left die even and right die even) = P(left die even) P(right die even)
= (1/2) (1/2) =1/4

- Pair of dice, what's the chance of rolling an "8"?

Event $E = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$
$P(E) = 5/36$

Be careful: $P(E) = |E|/|S|$ only **if** we are assuming the **uniform** distribution.

- Pair of dice, what's the chance of rolling an "8" if I tell you that both numbers came out even?

  Method 1: Imagine the new probability space:

  (2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)
  *** **** ***

  So probability is 3/9 = 1/3

  Method 2: A little more mechanically

  A = "rolled an 8"
  B = "both die are even"

  $P(A \mid B) = P(A \cap B)/P(B)$
  $= (3/36) / (9/36) = 1/3$

Eg 2. Back to the Poker examples

What's the probability space?
Sample space has $|S| = C(52,5)$
We can regard the points of S as **5-elements subsets**
of {2C,2D,2H,2S, 3C,3D,3H,3S,..., KC,KD,KH,KS, AC,AD,AH,AS} or {1, ...,52}
Probability measure is uniform: $P(a) = 1/|S|$.

Eg 3: Fair coin

Flip a fair coin 100 times.
What is the probability space?
$S = \{0,1\}^{100}$
$P(s) = 2^{-100}$ for all s in S.

What is the chance of getting **exactly 50 out of the 100 coin flips land heads?**

$P(50\text{Heads}) = C(100,50) / 2^{100} \approx 0.07959$ // note "100 choose 50" to Google
$P(51\text{Heads}) = C(100,51) / 2^{100} \approx 0.07803$

Eg 4: Biased coin

Now, what if the coin is biased?
Say that the coin lands **heads** with probability p**=** .51 and **tails** with probability $1-p$= .49.
each flip independent of the rest.

You flip the unfair coin 100 times. The coin lands heads a fraction p=0.51 of the time:

$S = \{0,1\}^{100}$ (same as before, but now)
$P(x) = p^{\#1(x)} (1-p)^{\#0(x)}$ where $\#1(x)$ = the number of 1-bits in the string x and
$\#0(x)$ = the number of 0-bits in the string x.

What's the Probability of 50 and 51 heads now?

$P(\text{50Heads}) = C(100,50)(.51\text{^}50)(.49\text{^}50) \approx 0.07801$
$\text{P(51Heads)} = C(100,51)(.51\text{^}51)(0.49\text{^}49) \approx 0.07906$

Makes sense -- 51 heads should now be the most likely number, and things should fall off from there. Before, 50 heads was the most likely outcome.

Eg 5: Birthday phenomenon

n=23 people gather in a room.
What' the chance that some two have the same birthday?
Assume nobody born 2/29, all other birthdays equiprobable.
$S = [1..365]^{23}$

P(SameBirthday) = 1 – Pr(AllBirthdaysDifferent)
= 1 – (1–1/365)(1–2/365) ... (1–22/365)

$$= 1 - \prod_{i=1}^{22} (1-1/i) \approx 0.507$$

That's as far as we got in lecture. See
http://en.wikipedia.org/wiki/Birthday_problem#Calculating_the_probability
if you didn't follow.