ECS 20 — Lecture 9 — Fall 2013 —24 Oct 2013

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Today:

o Sets of strings (languages)
o Regular expressions

Distinguished Lecture after class :
"Some Hash-Based Data Structures and Algorithms Everyone Should Know"
Prof. Michael Mitzenmacher, Harvard

Sets of STRINGS (elements of formal language theory)

Define and give examples:

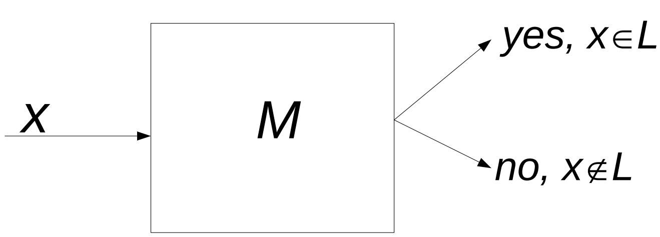
Alphabet – a finite nonempty set (of "characters") Σ
Strings - a finite sequence of characters drawn from some alphabet.
operation: **concatenation,** *xy* or *x* o *y*
Language – a set of strings, all of them over some alphabets
extend concatenation to langauges

Empty string (ε)
General set operations: union, intersection, complement; and concatenation
A^* -- Kleene closure – **define it** – $\cup_{i \geq 0} A^i$
$A^0 = \{ \varepsilon \}$ (why? For $A^i A^j = A^{i+j}$)
Σ*

BYTES = $\{0,1\}^8$
BYTES*

ENGLISH-WORDS = {a, aah, aardvark, aardwolf, aba, …, zymotic, zymurgy}
PRIMES = …
SAT = …

The relationship between languages and **decision questions**.



Regular expression over Σ:

1) a is a regular expression, for every $a \in \Sigma$. Also, symbols ε and ∅ are regular expressions.
2) I fα and β are regular expressions, then so are $(\alpha \circ \beta)$, $(\alpha \cup \beta)$, (α^*)

We routinely omit parenthesis, understanding it as a shorthand, with * binding most tightly, then concatenation, then union.

Example: a number
$(0\cup1\cup2\cup3\cup4\cup5\cup6\cup7\cup8\cup9)\ (0\cup1\cup2\cup3\cup4\cup5\cup6\cup7\cup8\cup9)^*$

A real number in decimal notation
$(0\cup1\cup2\cup3\cup4\cup5\cup6\cup7\cup8\cup9)(0\cup1\cup2\cup3\cup4\cup5\cup6\cup7\cup8\cup9)^* .\ (0\cup1\cup2\cup3\cup4\cup5\cup6\cup7\cup8\cup9)^*.$

An even number in binary
$(0\cup1)0$

Bit strings that start and top with the same bit (having at least one bit)
$00^* \cup 11^* \cup 0(0\cup1)^*0 \cup 1(0\cup1)^*1$

The complement of that set
$\varepsilon \cup 0(0\cup1)^*1 \cup 1(0\cup1)^*0$

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Exercise 1: write a regular expression for all strings over {0,1}
 that contain _some_ '111'.
       (0u1)* 111 (0u1)*

Exercise 2: write a regular expression for all strings over {a,b}
 whose length is divisible by 3.
       (aub)(aub)(aub))*


Exercise 3: write a regular expression for all strings over {a,b}
 whose length is NOT divisible by 3.
       (aub)(aub)(aub))*(aub) u
       (aub)(aub)(aub))*(aub)(aub)

Exercise 4: write a regular expression for all strings over {0,1}
       that contain an even # of 0's and an even # of 1's.
Kind of hard

Exercise 5: write a regular expression for all strings over {0,1}
       that contain the same number of 0's and 1's.
CAN'T BE DONE.  Why?  Take ECS120!
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Relations

(Change of topics. But do define some relations on strings, regular languages, and DFAs to tie the two topics together.)

DEF: A and B sets. Then a *relation* R is subset of A × B.

$R \subseteq A \times B$

Variant notation: x R y for $(x,y) \in R$

May use a symbol like ~ or < for a relations

x ~ y if $(x,y) \in \sim$

Relations in arithmetic, where A and B are both natural numbers:

= < <= > >=

| divides

what about succ, +, * **NO**: function symbols, not relations

In set theory:

$\in$

what about $\varnothing$ NO: constant symbol

Relations are useful for things other than numbers and sets and the like:

S = all UCD students for F13
C = all UCD classes for F13
P = all UCD professors for F13

E: enrolled relation $\subseteq$ S x C
s E c (ie, (s,c)\in E) - x is taking class y

T: teaches relation $\subseteq$C x P
c T p (ie, (c,p)\in T) - professor p is teaching class c this term

You can ***compose*** relations

what _*should*_
E o T

mean, do you think

E o T $\subseteq$ S × P S × C C × P -> S × P
s EoT p if there exists c in C such that s E c and c T p --
student s is taking some course that p is teaching --
p is s's teacher this term

What I've just given is the general definition

$R \subseteq X \times Y$

$S \subseteq Y \times Z$ then $R \circ S \subseteq X \times Z$ is $\{(x,z): \exists y \text{ in } Y\ xRy \text{ and } ySz\}$

What _should_ R^{-1} should be?
formalize

if $R \subseteq X \times Y$ is a relation that R^{-1} is the relation on $Y \times X$ where $(y,x)\ R^{-1}$ iff $x\ R\ y$.

More examples:

Often X = Y is the *same* set

Relations on natural numbers, real numbers, strings, etc.

X = set of strings

$x \le y$ "is a substring of y"

α and β are regular expressions.

$\alpha \sim \beta$ if $L(\alpha) = L(\beta)$

T/F: (0u1)^*(0u1)^* ~ (00 u 01 u 10 u 11)^* TRUE

e ~ e* TRUE

0(0u1)0 ~ 1(0u1)1 FALSE

Relations, continued. Let R be a relation on $A \times A$

We say that R is

Reflexive: if x R x for all x

Symmetric: if $x\ R\ y \rightarrow y\ R\ x$ for all x,y

Transitive: if x R y and $y\ R\ z \rightarrow x\ R\ z$ for all x, y, z

If R has all three properties, R is said to be an **equivalence relation**

	Reflexive	Symmetric	Transitive	comments
= on Integers (or anything else)	Yes	Yes	Yes	
<= , integers	Yes	No	Yes	antisymmetric
$\subseteq$, sets	Yes	No	Yes	*antisymmetric*
x E y if x and y are regular expressions and the regular L(x) = L(y)	Yes	Yes	Yes	blocks are languages
x S y if x is a substring of y	Yes	No	Yes	
x R y where x and y are strings and M is a some DFA and you go to the same state on processing	Yes	Yes	Yes	

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x and y

x | y  if 3 | x-y         Yes            Yes            Yes        Carefully
prove this one
                                                                   and write
out its blocks.
                                                                   Define when
n | m
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We only got to here – and then I jumped ahead to defining functions. We'll take up equivalence classes and quotients next time, as well as properties of functions, like injectivity and surjectivity.

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4. Functions
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Definition:  A function f is a relation on A x B such that
             there is one and only one pair a R b for every in A.

We write b=f(a) to mean that (a,b) in f.

(Just one way to do it: we could have defined functions as the primitive
and used the function to define the relation, putting in a pair
(a,f(a)) for every a in A.)

- We call A the domain of f,  Dom(f).
- We call B the *codomain* (or *target*) of f.
  Note that this does not mean the set {b: f(a)=b for some a in A}!
  That is a different (and important) st called the *Range* (or *image*)
  of f.  Denote it f(A).

Example 1:
 Domain={1,2,3}
 f(a) = a^2.
 Dom(f) = {1,2,3}
 f(A) = {1,4,9}
 co-domain: unclear, might be \N, might be \R, ....

Example 2:
 Domain = students in this class
 b(x) = birthdays, encoded as {1,..,12} x {1..31}.

 b(phil) = (7,31)
 b(ellen) = (4,1)

Example 3:
 f: \R -> \R defined by f(x) = x^2
 is it a function?
 Represent it as a graph

 Two functions f and g are equal, f=g, if their domains and ranges are equal
 and f(x) = g(x) for all x in Dom(f)

Function composition
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f o g
f: A -> B,   g: B -> C
the (g o f) : A -> C is defined by
(g o f)(x) = g(f(x))

Kind of "backwards" notation, but fairly tradition. Some algebrists
will reverse it,   (x) (f o g)   "function operates on the left"

Some computer scientists like to denote functions by "lambda expressions"
To say that f is the function that maps x to x^2 we write
f = lambda x. x^2
Here x is just a formal variable;
lambda x . x^2   =   lambda y . y^2
The domain is not explicitly

Functions don't have to be defined on numbers, of course

|x| = maps \Sigma^* -> \N
hd(x) = the first character of the string x, x\ne emptystring
tl(x) = all but the first character of x (define how when x=\emptystring)?
dim(A) = the dimensions of the matrix A, regarded as a pair of natural
numbers