ECS 20 – Fall 2021 – P. Rogaway Probability

Def: A (finite) **probability space** (S, P) is

- a finite set S (*the sample space*) and
- a function μ: $S \to [0,1]$ (the *probability measure*) such that

$$\sum_{x \in S} \mu(x) = 1$$

(alternative notation: Ω for S, P *for* μ)

In general, whenever you hear *probability* make sure that you are clear **what** is the probability space is: what is the sample space and what is the probability measure on it.

An **outcome** is a point in S.
An **event** $A \subseteq S$ is a subset of S.

Def: Let A be an event of probability space (S, P).

$$P(A) = \sum_{a \in A} \mu(a)$$ // The notation Pr is common, too

The probability of event A. $P(\emptyset)=0$.

Def: The **uniform** distribution is the one where $P(a) = 1/|S|$ —i.e., all points are equiprobable.

Def: Events A and B are **independent** if $P(A \cap B) = P(A)\, P(B)$.

Def: $P(A \mid B) = P(A \cap B) / P(B)$ assuming $B \neq \emptyset$

Def: A **random variable** is a function X: $S \to \mathbb{R}$ from the sample space to the reals. // Sometimes we allow a different codomain.
// Sometimes we use a special font for RVs, like X

Def: $$E[X] = \sum_{a \in S} P(a)\, \mu(a)$$ // **expected value** of X ("average value")
// Alternatively write $E(X)$ or $\mathbf{E}[X]$

Conditional probability: Used to capture prior knowledge.
Def: $P(A \mid B) = P(A \cap B) / P(B)$. *This requires* $B \neq \emptyset$!

Propositions:

- $P(\emptyset) = 0$ and $P(S) = 1$ (by definition)

- $P(A) + P(A^c) = 1$, or $P(A) = 1 - P(A^c)$

- If A and B are disjoint events (that is, disjoint sets) then
  $P(A \cup B) = P(A) + P(B)$

- **Sum bound:**
  $P(A \cup B) \leq P(A) + P(B)$

- **Principle of inclusion-exclusion:**
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

- $P(A) = P(A \mid B)\, P(B) + P(A \mid B^c)\, P(B^c)$

- $\mathbf{E}(X+Y) = \mathbf{E}(X) + \mathbf{E}(Y)$ // expectation is linear.

Dice

The singular, the students assure me, is *die*.
Just like the singular of *mice* and *mie*.

Let's identify the probability spaces involved.

- You roll a fair die one time:
  $S = \{1,2,3,4,5,6\}$
  $\mu(1) = \mu(2) = \ldots = \mu(6) = 1/6$

  "you roll an even number" is an event.
  Event is $A = \{2,4,6\}$. $P(A) = 3 * (1/6) = 1/2$.

- Pair of dice. You roll a pair of dice. The die are distinct.
  $S = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$
  $\mu((a, b)) = 1/36$ for all $(a, b) \in S$

Illustrate independence.

P(left die even and right die even) = P(left die even) P(right die even)
= (1/2) (1/2) =1/4

Problem 1. You role a pair of dice, what's the probability of rolling an "8"?

Event E = {(2,6), (3,5), (4,4), (5,3), (6,2)}
$P(E)$ = 5/36

Be careful: $P(E) = |E|/|S|$ only if we are assuming the uniform distribution.

Problem 2. Your roll a pair of dice. What's the chance of rolling an "8" if I tell you that both numbers came out even? [in Monday's class we did sum of numbers was even]

Method 1: Imagine the new probability space:

(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)
*** *** ***

So probability is 3/9 = 1/3

Method 2: A little more mechanically

A = "rolled an 8"
B = "both die are even"

$P(A \mid B) = P(A \cap B) / P(B)$
= (3/36) / (9/36) = 1/3

Poker examples

Problem 3. What's the probability space and probability measure for dealing a poker hand?
Sample space has $|S| = C(52,5)$
We can regard the points of S as 5-elements subsets of
Cards = {2,3,4,5,6,7,8,9,J,Q,K,A} x {c,d,h,s}

Probability measure is uniform: $\mu(a) = 1/|S|$.

Fair coin

Problem 4. You flip a fair coin 100 times.
What is the probability space and probability measure?

$S = \{0,1\}^{100}$

$\mu(a) = 2^{-100}$ for all $a \in S$.

Problem 5. You flip a coin 100 times. What is the chance of getting exactly 50 out of the 100 coin flips land heads?

$P(50\text{Heads}) = C(100,50) / 2^{100} \approx 0.07959$

$P(51\text{Heads}) = C(100,51) / 2^{100} \approx 0.07803$

Biased coin

Now, what if the coin is biased?
Say that the coin lands **heads** with probability $p = .51$ and **tails** with probability $1 - p = .49$.
each flip independent of the rest.

Problem 6. You flip a coin 100 times. The (unfair) coin lands heads a fraction $p = 0.51$ of the time. Now what's the chance of getting 50 heads? 51 heads?

$S = \{0,1\}^{100}$ (same as before, but now)

$\mu(x) = p^{\#1(x)} (1-p)^{\#0(x)}$

where $\#1(x)$ = number of 1-bits in the string x and
$\#0(x)$ = the number of 0-bits in the string x.

What's the probability of 50 and 51 heads now?

$P(50\text{Heads}) = C(100,50)(.51^{50})(.49^{50}) \approx 0.07801$

$P(51\text{Heads}) = C(100,51)(.51^{51})(0.49^{49}) \approx 0.07906$

Makes sense -- 51 heads should now be the most likely number, and things should fall off from there. Before, 50 heads was the most likely outcome.

Expectations

Again:

Def: A **random variable** is a function $X\colon\ S \to \mathbb{R}$ from the sample space to the reals. // Sometimes we allow a different codomain.
// Sometimes we use a special font for RVs, like X

Def: $E[X] = \sum_{a \in S} P(a)\, \mu(a)$ // **expected value** of X ("average value")
// Alternatively write $E(X)$ or $\mathbf{E}[X]$

Problem 7. A standardized multiple choice test with 5 answers per question takes off ¼ point for each wrong answer, gives 1 points for each right question. Maples guesses on question 5. What is her expected score on the problem?

Problem 8. Alice rolls a die. What do you expect the square of her roll to be?

Could be 1 .... could be a 36.

Definition: $E[X] = \sum_{a} X(a)\, \mu(a)$

So, in this problem,

$E[X] = 1(1/6) + 2^2(1/6) + 3^2(1/6) + ... + 6^2(1/6)$
$= 1/6(1+4+9+15+25+36)$
$= 91/6$
≈ 15.2

Conditional probability uses: $P(A \mid B) = P(A \cap B) / P(B)$ assuming $B \neq \emptyset$

Proposition: $P(A) = P(A \mid B)\, P(B) + P(A \mid B^c)\, P(B^c)$

$$\frac{P(A\cap B)}{P(B)}\; P(B) \;+\; \frac{P(A\cap B^c)}{P(B^c)}\; P(B^c)$$

Birthday phenomenon

n=23 people gather in a room.
What's the chance that some two have the same birthday?
Assume nobody born 2/29, all other birthdays equiprobable.
$S = [1..365]^{23}$ **Named events:**
D_1 = entire sample space
D_2 = people 1,2 have distinct birthdays
D_3 = people 1,2,3 have distinct birthdays
…
D_{23} = people 1, 2, …, 23 have distinct birthdays
C_{23} = some two people among 1, 2, …, 23 have the same birthdays

$P[D_{23}] = P[D_{23} \mid D_{22}]\, P[D_{22}]$ ~~$+ P[D_{23} \mid !D_{22}]\, P[!D_{22}]$~~

$= P[D_{23} \mid D_{22}]\, P[D_{22} \mid D_{21}]$

$= P[D_{23} \mid D_{22}]\, P[D_{22} \mid D_{21}]\, P[D_{21} \mid D_{20}]$

…

$= P[D_{23} \mid D_{22}]\, P[D_{22} \mid D_{21}] \ldots P[D_2 \mid D_1]$

$= (1 - 22/365)(1-21/365) \ldots 1(1-1/365)\, 1$

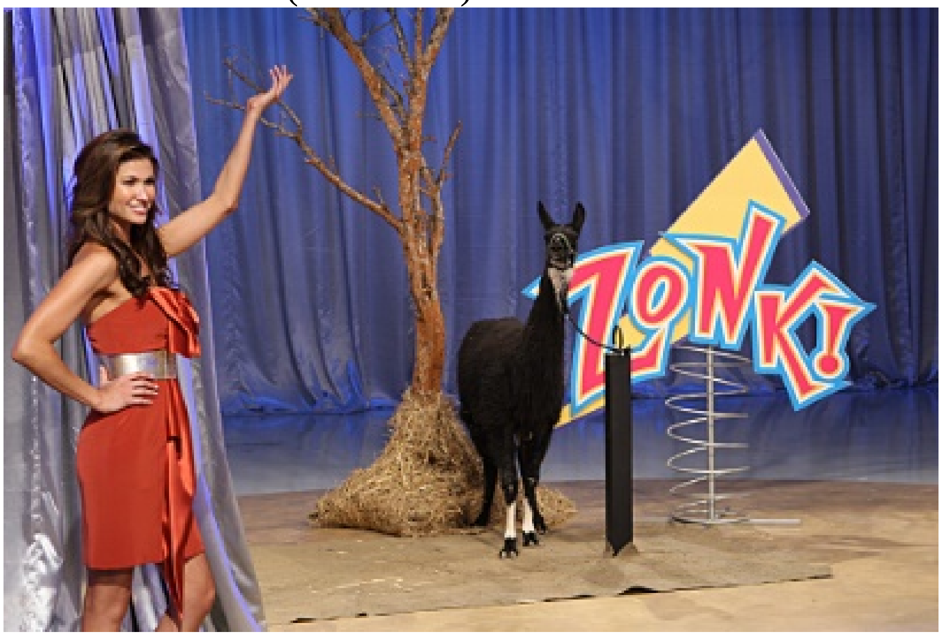
$$= \frac{343}{365}\;\frac{344}{365}\;\frac{345}{365}\;\ldots\;\frac{365}{365}$$

$= (1/365)^{23}\,(365\; 364 \ldots 343)$
≈ 0.493

$P(C_{23}) = 1 - \Pr(D_{23})$
$= 1 - (1-1/365)(1-2/365) \ldots (1-22/365)$
$\approx 1 - 0.493$
$= 0.507$

Monty Hall Problem

Let's make a Deal (1963-1968)




```
 ========     ========     ========
 |      |     |      |     |      |
 | bad  |     | bad  |     | good |
 |      |     |      |     |      |
 ========     ========     ========
    1            2            3
```

A good prize is hidden behind a random curtain/door (junk, "zonk", behind the other two).
You choose an arbitrary door. The host opens one of the unselected doors that does NOT contain the good prize. Should you switch to the other door?

loc of good prize   which unselected door to open if host must choose
$S = \{1, 2, 3\} \times \{0, 1\}$ not really relevant

WIN = event that the contestant gets the good prize

Strategy STICK: choose door 1 and stick with it: P (WIN)=1/3

Strategy SWITCH: choose door 1 and then switch (always) when offered a chance.

Calculation by method 1:

(1,0)	(2,0)	(3,0)	Second bit doesn't matter.
Lose	Win	Win	
(1,1)	(2,1)	(3,1)	
Lose	Win	Win	

P(Win) = 4/6 = 2/3

Calculation by method 2:

$$P(\text{Win}) = P(\text{Win} \mid \text{initialCorrect})\ P(\text{initialCorrect}) + P(\text{Win} \mid \text{initialIncorrect})\ P(\text{initialIncorrect})$$

$$= 0 + 1\ (2/3) = 2/3$$