

## Problem Set 10 — Due Thursday, December 2, at 11 am

There are more questions on this problem set than usual, yet the first 8, which are counting exercises, should be answerable pretty quickly, and require very little text. For those problems please provide your answers in **two different forms**: (1) a formula involving only constants,  $P(n, k)$  values,  $C(n, k)$  values, factorials, and basic arithmetic operations; and (2) an explicit number, determined by evaluating (1). An approximate value is fine if the exact value is extremely large, like  $10^{20}$  or more. Being explicit is fun, and also makes things easier to grade.

- In how many ways can ten boys and four girls sit down in a row?
  - In how many ways can they sit in a row if the boys sit together and the girls sit together?
  - In how many ways can they sit in a row if the girls sit together?
  - In how many ways can they sit in a row if the girls sit together and the boys surround them (annoying creatures that they are)?
- Four cats and five mice enter a race. The mice are clearly superior; they place first, second, and third. In how many ways can this happen?
- How many permutations over the letters **a, b, c, d, e, f, g** contain neither the pattern **bge** nor the pattern **eaf**?
- (Leading zeros are not permitted in either part of this question)
  - How many seven-digit numbers have no repeated digits?
  - How many seven-digit numbers with no repeated digits contain a 3 but not a 6?
- A group of people is comprised of six from California, seven from Oregon, and eight from Washington.
  - In how many ways can a committee of six be formed with two people from each state?
  - In how many ways can a committee of seven be formed with at least two people from each state?
- Eve has ten apples, and plans to give at most three of them to Adam. How many ways can she do this?
- How many five-card hands dealt from a standard deck of 52 playing cards are all of the same suit? If you deal out a random hand, what is the probability that it will have this property?
- A woman has nine close friends.
  - In how many ways can she invite six of them to dinner?
  - Repeat (a) if two of her friends are divorced from one another and mustn't both be invited.
  - Repeat (a) if the friends consist of three single people and three married couples and if a husband or wife is invited, the spouse must be invited, too.
- Prof. Rogaway asks ten true/false questions. The ornery professor assigns grades of: 10 points for each correct response, 0 points for each absent response, and  $-10$  points for each incorrect response—but where negative totals are always replaced by zero. Poor Ziggy understands nothing of Rogaway's class, so flips a fair coin to decide if he will answer each question true or false (he leaves no question unanswered).

- (a) For each number of points  $N$ , what is the probability that Ziggy will earn  $N$  points?
  - (b) What is the expected number of points  $E[X]$  that Ziggy will earn? This is just the sum, over all possible number of points earned, of that number of points times the probability of earning it.
  - (c) Is Ziggy wise to guess (as opposed to leaving things blank)?
10. Five misanthropes (all computer science professors) live on a triangular island of Australasia. The island's dimensions are 2 miles  $\times$  2 miles  $\times$  2 miles. Show that some two of the misanthropes must live within a mile of one another. (They won't be happy about it.) Hint: the *pigeonhole principle* says that if  $N$  items are placed into  $n$  bins, where  $n < N$ , then some bin must contain two or more items.
11. Prove that for any positive number  $n$ , there will be a nonzero multiple of  $n$  whose digits, base-10, are all 0s and 1s. Hint: *Pigeons* 1, 11, 111, ...