ECS-20 Section A (MWF) – Fall 2021

First name: _________________________ LAST NAME: _________________________
Room #: ___________________________ Seat #: _________________________________

“I attest that I have in no manner cheated on this exam.”
Signature: __________________________ SID: ______________________________

Instructions

(1) The exam has five numbered pages, not including this one.
(2) You may not sit next to anyone you know.
(3) No calculators, phones, or smartwatches. Devices must be powered off and put in your bag.
(4) You may have one page of notes, 8.5 × 11, printed on one-side. Put your name on it and turn it in with your exam.
(5) Read each question carefully and write each answer neatly.
(6) If you don’t understand some question or notation, you can ask.
(7) On the True/False, if you don’t know, guess. Your score, prior to rounding to the nearest natural number, will be linearly proportional to the number of correct answers.
(8) When an answer is a number, you can name it explicitly or provide an expression involving basic arithmetic, factorials, \( C(n, k) \), or \( P(n, k) \).

<table>
<thead>
<tr>
<th>On page</th>
<th>you got</th>
<th>out of</th>
</tr>
</thead>
<tbody>
<tr>
<td>1–2</td>
<td></td>
<td>120</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>60</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>60</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>60</td>
</tr>
<tr>
<td>Σ</td>
<td></td>
<td>300</td>
</tr>
</tbody>
</table>
1 True / False

Darken the correct answer. If you don’t know, guess.

1. If $P$ is False then $P \rightarrow Q$ is True. True False

2. The XOR of 80 Boolean variables will be 1 if half of them are 1. True False

3. $P \rightarrow (Q \rightarrow R) \equiv (P \rightarrow Q) \rightarrow R$ True False

4. $|=(X_1 \leftrightarrow X_2)(X_2 \leftrightarrow X_3) \rightarrow (X_1 \leftrightarrow X_3)\)$. True False

5. It is possible to realize a NOR gate using only XOR gates. True False

6. If $a, b$, and $x$ are bits and $a \oplus x = b$ then $x = a \oplus b$. True False

7. If $\phi$ is a CNF formula then $(\neg \phi)$ is a DNF formula. True False

8. $(\exists n)(P(n) \land P(n+1) \land n > 2) \equiv (\forall n)(P(n) \lor P(n+1) \lor n \leq 2)$ True False

9. For any sets $A$, $B$, and $C$, if $A \times B = A \times C$ then $B = C$. True False

10. The two’s complement representation of $-2$ as an 8-bit number is 11111110. True False

11. $A \subseteq \mathcal{P}(A)$ for any set $A$. True False

12. The set of 32-bit numbers forms a group under addition mod $2^{32}$. True False

13. The group $S_{10}$ of permutations on $\{1, 2, \ldots, 10\}$ has $2^{10}$ points. True False

14. For any relation $\sim \subseteq X \times X$, if $a \sim b$ and $b \sim c$ then $a \sim c$. True False

15. There is an injective function $f: \mathbb{N} \rightarrow \mathbb{R}$. True False

16. There is an injective function $f: \mathbb{R} \rightarrow \mathbb{N}$. True False

17. If $f: A \rightarrow B$ and $g: B \rightarrow C$ are bijective then $g \circ f: A \rightarrow C$ is bijective. True False
18. If \( A \) is a finite set then \( |A \times A| = |A|^2 \)
   - True  False

19. \( f(n) = 2n \mod 5 \) is a bijection on \( \mathbb{Z}_5 = \{0, 1, 2, 3, 4\} \).
   - True  False

20. Every language \( L \subseteq \{0, 1\}^* \) can be decided by some computer program.
   - True  False

21. If \( L \) is a finite language then \( |L \circ L| = |L|^2 \).
   - True  False

22. If a language \( L \) contains a string of length 5 then \( L^* \) is infinite.
   - True  False

23. \( \{00, 01, 10, 11\}^* = \{0, 1\}^* \)
   - True  False

24. There is a BNF\(^1\) description of any finite language.
   - True  False

25. Algorithm D runs in \( 2^{80} \) steps, each taking 1 machine cycle. You need to run it once a day. Doing so should be feasible on a fast, recent computer.
   - True  False

26. If \( f(n) \in O(n) \) and \( g(n) \in O(n) \) then \( (f(n) + g(n) + \log(n)) \in O(n) \).
   - True  False

27. \( \sum_{i=1}^{n} i^2 \in \Theta(n^2) \).
   - True  False

28. \( C(n, 2) \in \Theta(n^2) \).
   - True  False

29. \( P(n, k) = P(n, n - k) \) for all \( 1 \leq k \leq n \).
   - True  False

30. There are more than 100 ways to rearrange the letters of \textit{zebra}.
   - True  False

31. To talk about the probability of being dealt various poker hands we use a probability space \((S, \mu)\) with a sample space \( S \) consisting of \( |S| = 52 \) cards.
   - True  False

32. For any blockcipher \( E : \{0, 1\}^{256} \times \{0, 1\}^{128} \to \{0, 1\}^{128} \), \( E(K, X) = E(K', X') \) implies \( (K, X) = (K', X') \).
   - True  False

33. Graph \( K_{10} \), the clique of size 10, is Eulerian\(^2\)
   - True  False

34. Graph \( K_{10} \), the clique of size 10, is 2-colorable (bipartite).
   - True  False

---

\(^1\) Backus Normal form, as in PS #3, problem 5.

\(^2\) Recall that \( K_n \) has \( n \) vertices and an edge between each pair of them; and that a graph is Eulerian if there is a cycle that passes through each of its edges once and only once.
2 Short Answer

If you see a box, make sure your final answer lives in it. Don’t worry; it won’t feel claustrophobic.

1. \( \text{gcd}(860, 200) = \) 

2. How many (positive) divisors \( d(n) \) does \( 300125 = 5^3 \cdot 7^4 \) have? 

3. Explicitly name the elements: \( \{1, 2\} \times \{3, 4\} = \) 

4. How many functions are there from \( \{0, 1\}^3 \) to \( \{1, 2, 3\} \)? 

5. Neatly write a truth table for the Boolean function \( A \rightarrow (B \rightarrow C) \). Order the rows in the conventional way. Use 0 and 1 to represent false and true.

6. Draw a circuit that computes the XOR function, \( y = a \oplus b \), limiting yourself to only AND, OR, and NOT gates. Don’t use more gates than necessary. Note: If you forgot the symbol for a gate just draw a box and label it AND, OR, or NOT.
7. Suppose I’ve defined a function \( f : \mathbb{N} \to \mathbb{N} \) and I want to show that \( f(n) \leq 10n \) for all \( n \geq 5 \). To do this by induction it suffices to prove that (the basis) and that (the inductive step).

8. A woman has ten close friends. In how many ways can she invite five of them to dinner if two of them are inseparable and must either both be invited, or neither one invited?

9. An urn contains 15 red balls and 10 white balls. Five random balls are removed. What is the probability that all of them are red?

10. What exactly would you enter in LaTeX to typeset the follow: Let \( f(n) = n^2 \log B_n \)

11. Consider the recurrence: \( T(n) = T(n/2) + 1 \). Assume \( T(n) = 1 \) when \( n \leq 1 \). Solve the recurrence to within a \( \Theta(\cdot) \) bound: \( T(n) \in \) .

12. Define the equivalence relation \( \sim \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+ \) by: \( a \sim b \) if \( a \) and \( b \), written as decimal numbers, have the same first (leftmost) digit. This equivalence relation has equivalence classes and the three smallest elements in one of them are: Note: \( \mathbb{Z}^+ = \{1, 2, 3, \ldots\} \).
13. How many ways can you paint the edges of $K_{10}$ with each edge red or blue?

14. Where did you encounter the following?

Twenty random cards are placed in a row, all face-down. A move consists of turning a face-down card face-up, and turning over the card immediately to the right. Show that no matter what the choice of cards to turn, this sequence of moves must terminate.


Suppose for contradiction that $S$ were countable. Then there would be some enumeration of it, a list $X_1, X_2, X_3, \ldots$ that includes all the superstrings.


Is $B$ in the list $X_1, X_2, X_3, \ldots$? If so then $B = X_j$ for some particular $j \in \mathbb{N}$. Yet it can’t be the case that $B = X_j$ because

That $B$ is not in $X_1, X_2, X_3, \ldots$ contradicts that list containing all superstrings.

16. How many strings can you get by rearranging the characters of **aardwolf**?

Note: if you flip the first two letters, say, it’s not a new string.

17. In the “Monty Hall” (“Let’s Make a Deal”) problem from our final lecture, what is a contestant’s probability to select the correct door assuming she always does switch doors when the host offers her the opportunity?

18. **Complete the following proof that some positive multiple of 159 has digits that are all 0s and 1s. Be succinct.**

For $i \in [1..160]$, let $A_i = \overbrace{111 \cdots 111}^i$ and let $a_i = A_i \mod 159$.

By the