Today:
- Review
- More counting
- Naïve probability

Review

\[ 2^n = \text{Number of subsets of } n \text{ items} \]
\[ = \text{number of } n\text{-bit binary strings} \]
\[ = \text{number of ways to paint } n \text{ items with 2 different colors.} \]

\[ d^n = \text{Number of length-} n \text{ strings over an alphabet of } d \text{ character} \]
\[ = \text{number of ways to paint } n \text{ items with } d \text{ different colors.} \]

\[ n! = \text{Number of ways to arrange } n \text{ different items} \]
\[ = \text{Number of ways to order } \{1,2,...,n\} \]

\[ P(n, k) = \text{The number of ways to arrange } k \text{ items drawn, without} \]
\[ \text{replacement, from a universe of } n \text{ items} \]
\[ = \text{Number of ways to fill } k \text{ bins, one item per bin,} \]
\[ \text{from a universe } \{1,...,n\} \]
\[ = n(n - 1) ... (n - k + 1) \]
\[ = n!/(n - k)! \]

\text{No replacement; an item, once used, is gone.}

\[ C(n, k) = \text{Number of ways to fill a bin with } k \text{ items from a} \]
\[ \text{universe } \{1,...,n\} \]
\[ = \text{number of } k\text{-element subsets from a set of } n \text{ different items} \]
\[ = n! / k!(n - k)! \]
\[ = P(n, k)/k! \]

\text{No replacement; an item, once used, is gone.}

Supported by Google's search-line calculator as in “100 choose 50”
Alternate notation: \( \binom{n}{k} \)

\[ C(n, 2) = \text{Number of 2-element subsets from an n-element set} \]
\[ = \text{number of } k\text{-element subsets from a set of } n \text{ different items} \]
\[ = \frac{n(n-1)}{2} \]

**Product rule** = if event \( A \) can occur in \( a \) ways and, independent of this, event \( B \) can occur in \( b \) ways
then the number of combinations of ways for \( A \) and \( B \) to occur is \( ab \).

\[ P(n, k) = n(n-1) \cdots (n-k+1) / n! \]
\[ = \frac{n(n-1) \cdots (n-k+1)}{(n-k)!} \]

**Sum rule** = if event \( A \) can occur in \( a \) ways and event \( B \) can occur in \( b \) ways,
but both events cannot occur together,
then the number of ways for \( A \) or \( B \) to occur is \( a+b \).
Really just a statement that $|A \cup B| = |A| + |B|$ for disjoint $A, B$.

**Inclusion/exclusion counting:**

$|A \cup B| = |A| + |B| - |A \cap B|$

And generalizations, like

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

**Reminder:** $\log (n!) \approx n \log n$

**Review**

**Problem 8.** In how many ways can 10 adults and 5 children be positioned in a line so that no two children are next to each other?

**Answer:** $10! \times P(11,5) = 10! \times \frac{11!}{6!} = 201,180,672,000 \approx 10^{11.3}$

**More Examples**

**Problem A.** Ten members of a club line up for a photograph. The club has **one president**, **one VP**, **one secretary**, and **one treasurer**.

- How many ways are there to line up the ten people?
  
  $10!$

- How many ways are there to line up the ten people if the VP must be beside the president in the photo?
  
  $2 \times 9!$

- How many ways are there to line up the ten people if the president must be next to the secretary and the VP must be next to the treasurer?
  
  $2 \times 2 \times 8!$
**Problem B.** You toss a coin 8 times. How many ways can the coin tosses land? How many ways with 5 heads total?

**Answer:** \( C(8,5) = 56 \)

Note this is \( C(8,5) = C(8,3) \).

In general, \( C(n, k) = C(n, n-k) \).

Also notice that \( 2^8 = C(8,0) + C(8,1) + \ldots + C(8,8) \).

In general, \( 2^n = C(n,0) + C(n,1) + \ldots + C(n, n) \)

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**Problem D.** How many 6-element subsets are there of the letters A ... Z?

\( C(26,6) = 230,230 \)

How many 2-element subsets are there of the letters A ... Z?

\( C(26,2) = 26*25/2 = 325 \). In general, \( C(n,2) = n(n-1)/2 \)
Are there more 2-element subsets or 24-element subsets?
\[ \text{The same.} \]

Are there more 2-element subsets or 3-element subsets?
\[ \text{More 3-element subsets. } \frac{(n-2)}{3} = 8 \text{ times as many.} \]

How many subsets all together are there of are there of A ... Z?
\[ 2^{26} = 67,108,864 \]

**Problem E.** An urn contains 15 red, distinctly numbered balls, and 10 blue, distinctly numbered balls.
5 balls are removed.

(A) How many different samples are possible?
\[ \text{Answer: } C(25,5) = 53,130 \]

(B) How many samples contain only red balls?
\[ \text{Answer: } C(15,5) = 3003. \]

(B') So what is the **probability** that a random sample will contain only red balls?
\[ \text{Answer: } \frac{3003}{53,130} \approx 0.05652 \ (05.652 \%) \ (a \ little \ more \ than \ about \ 1 \ in \ 18) \]

(C) How many samples contains 3 red balls and 2 white balls?
\[ \text{Answer: } C(15,3) \times C(10,2) = 20,475 \]

(C') So what's the probability that a random sample will contain 3 red balls and two blue ball?
\[ \text{Answer: } \frac{20,475}{53,130} \approx 0.3854 \ (38.54\%) \]
**Problem F.** How many numbers are there between 1 and 1000 have are not divisible by 3, 5, or 7

\[ A_3 = \text{numbers in } [1..1000] \text{ that are divisible by } 3. \ |A_3| = 333 \]
\[ A_5 = \text{numbers in } [1..1000] \text{ that are divisible by } 5. \ |A_5| = 200 \]
\[ A_7 = \text{numbers in } [1..1000] \text{ that are divisible by } 7. \ |A_7| = \left\lfloor \frac{1000}{7} \right\rfloor = 142 \]

\[ A_{3,5} = \text{numbers in } [1..1000] \text{ that are divisible by } 3 \& 5. \ |A_{3,5}| = \left\lfloor \frac{1000}{15} \right\rfloor = 66 \]
\[ A_{5,7} = \text{numbers in } [1..1000] \text{ that are divisible by } 5 \& 7. \ |A_{5,7}| = \left\lfloor \frac{1000}{35} \right\rfloor = 28 \]
\[ A_{3,7} = \text{numbers in } [1..1000] \text{ that are divisible by } 3 \& 7. \ |A_{3,7}| = \left\lfloor \frac{1000}{21} \right\rfloor = 47 \]
\[ A_{3,5,7} = \text{nums in } [1..1000] \text{ that are divisible by } 3 \& 5 \& 7. \ |A_{3,5,7}| = \left\lfloor \frac{1000}{3 \times 5 \times 7} \right\rfloor = 9 \]

So answer, by inclusion/exclusion, is \[ 1000 - 333 - 200 - 142 + 66 + 28 + 47 - 9 = 457 \]

**Problem G. Poker.** Deck of 52 cards, these having 13 “values” and 4 suits. 5 cards are dealt. We are interested in the probability of being dealt certain kinds of hands.

royal flush = 10JQKA of one suit.
straight flush = five consecutive cards: 2345, , ..., , 10JQKA in any suit.
four of a kind = four cards of one value (e.g., all four 9's)
full house = 3 cards of one value, 2 cards of another value. (Eg, 3xA, 2x4).
flush = five cards of a single suit
three of a kind = 3 cards of one value, a fourth card of a different value, and a fifth card of a third value
two pairs = two cards of one value, two more cards of a second value, and the remaining card of a third value
one pair = two cards of one value, but not classified above

a) How many poker hands are there?

Answer: \( C(52,5) = 2,598,960 \)

b) How many poker hands are full houses?

Answer: A full house can be partially identified by a pair, like (J,8), where the first component of the pair is what you have three of, the second component is what you have two of. So there are \( P(13,2) = 13 \times 12 \) such pairs.
For each there are \( C(4,3) = 4 \) ways to choose the first component, and \( C(4,2) = 6 \) ways to choose the second component. So all together there are \( 13 \times 12 \times 4 \times 6 = 3,744 \) possible full houses.

c) What’s the probability of being dealt a full house?

\[
3,744 / 2,598,960 \approx 0.001441 \approx 0.14\%
\]

\[ P[\text{FullHouse}] \approx 0.001441 \]

The probability of an event is a real number between 0 and 1 (inclusive). If asked what’s the probability of something, don’t answer with a “percent”, and don’t answer with something outside of \([0,1]\). When we give something in “percent’s”, we are giving a probability multiplied by 100.

d) How many poker hands are two pairs?

Answer: We can partially identify two pairs as in \{J, 8\}. Note that now the pair is now unordered. There are \( C(13,2) \) such sets. For each there are \( C(4,2) \) ways to choose the larger card and \( C(4,2) \) ways to choose the smaller card. There are now 52 - 8 = 44 remaining cards one can choose as the fifth card (to avoid a full house, there are 8 “forbidden” cards). So the total is

\[
C(13,2) \times C(4,2) \times C(4,2) \times 44 = 123,552.
\]

e) What is the probability of being dealt two pairs?

\[
C(13,2) \times C(4,2) \times C(4,2) \times 44 / C(52,5) = 123,552 / 2,598,960 \\
\approx 0.047539 \approx 4.75\%
\]

\[ P[\text{TwoPairs}] 0.047539 \]

Problem H. How many different passwords are there that contain only digits and lower-case letters and satisfy the given restrictions?

(a) Length is 6 and the password must contain at least one digit.

All length-6 passwords – Those with only letters
\[ = 36^6 - 26^6 = 1867866560 \approx 2^{31} \]
(a) Length is 6 password that contain **at least one digit** and **at least one letter**.

All length-6 passwords – Those with only letters - Those with only digits
\[= 36^6 - 26^6 - 10^6 = 1866866560 \approx 2^{31}\]

**Problem 1.** A 5-card hand is drawn from a deck of standard playing cards.

(a) How many **5-card hands** have **at least one club**?

Total # Of Deals – Number Of Deals With No Clubs
\[= C(52,5) – C(39,5) = 2023203\]

(b) How many 5-card hands have **at least two cards** with the same rank?

Total # Of Deals – # of Deal where All 5 cards have different ranks
\[= C(52,5) – C(13,5) \times 4^5 = 1281072\]